Homework #4

Problem 3:

We need to find the expression for $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ in the system of coordinates $(r, \cos\theta, \phi)$. There are two ways of solving this problem:

1) We can use the expression given in class:

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\delta(r_1 - r_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\phi_1 - \phi_2)}{|J(x, y, z; r, \cos\theta, \phi)|} \tag{1}$$

We know that $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$; then the Jacobian is given by:

$$J = \begin{vmatrix} \sin\theta\cos\phi & -r\tan\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & -r\tan\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & r & 0 \end{vmatrix}, \tag{2}$$

The determinant gives $-r^2$ (use that $\frac{d(\sin\theta)}{d(\cos\theta)} = \frac{d[(1-x^2)^{1/2}]}{dx} = \frac{-x}{(1-x^2)^{1/2}} = -\tan\theta$). Then:

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\delta(r_1 - r_2)\delta(\cos\theta_1 - \cos\theta_2)\delta(\phi_1 - \phi_2)}{r_1^2}. \tag{2}$$

Notice that since $\delta(r_1 - r_2)$ is symmetric we can use either r_i, depending on the integration variable that we use, in the denominator.

2) The second form is using the fact that the delta integrated over all space is equal to 1. Then:

$$1 = \int_0^\infty A\delta(r_1 - r_2) r_1^2 dr_1 \int_{-1}^1 \delta(\cos\theta_1 - \cos\theta_2) d(\cos\theta_1) \int_0^{2\pi} \delta(\phi_1 - \phi_2) d\phi_1. \tag{3}$$

We see that the integrals over ϕ_1 and over $\cos\theta_1$ give 1. In order to get the integral over r_1 to be equal to 1, we need to set $A = 1/r_1^2$ and we obtain the same result as in Eq.(2).