Homework #4

**Problem 4**:

a)

$$A^{ij} = \begin{pmatrix} V^1W^1 & V^1W^2 \\ V^2W^1 & V^2W^2 \end{pmatrix}, \tag{1}$$

$$A_{ij} = \begin{pmatrix} V_1W_1 & V_1W_2 \\ V_2W_1 & V_2W_2 \end{pmatrix}, \tag{2}$$

$$A^i{}_j = \begin{pmatrix} V^1W_1 & V^1W_2 \\ V^2W_1 & V^2W_2 \end{pmatrix}, \tag{3}$$

$$A_i{}^j = \begin{pmatrix} V_1W^1 & V_1W^2 \\ V_2W^1 & V_2W^2 \end{pmatrix}, \tag{4}$$

Notice that in system S, $V_i = V^i$ and $W_i = W^i$ because it is a cartesian system.

b) Now we need to find the transformed of the tensors in S' and since S' is not a cartesian system we need to follow the transformation rules for covariant and contravariant components:

$$A'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}A^{kl} = U^i{}_kU^j{}_lA^{kl}, \tag{5}$$

$$A'^{11} = U^1{}_1U^1{}_1A^{11} + U^1{}_1U^1{}_2A^{12} + U^1{}_2U^1{}_1A^{21} + U^1{}_2U^1{}_2A^{22} = A^{11} - (A^{12} + A^{21})\text{cotan}\alpha + A^{22}(\text{cotan}\alpha)^2. \tag{6}$$

$$A'^{12} = U^1{}_1U^2{}_1A^{11} + U^1{}_1U^2{}_2A^{12} + U^1{}_2U^2{}_1A^{21} + U^1{}_2U^2{}_2A^{22} = \frac{A^{12}}{\sin\alpha} - A^{22}\frac{\cos\alpha}{\sin^2\alpha}. \tag{7}$$

$$A'^{21} = U^2{}_1U^1{}_1A^{11} + U^2{}_1U^1{}_2A^{12} + U^2{}_2U^1{}_1A^{21} + U^2{}_2U^1{}_2A^{22} = \frac{A^{21}}{\sin\alpha} - A^{22}\frac{\cos\alpha}{\sin^2\alpha}. \tag{8}$$

$$A'^{22} = U^2{}_1U^2{}_1A^{11} + U^2{}_1U^2{}_2A^{12} + U^2{}_2U^2{}_1A^{21} + U^2{}_2U^2{}_2A^{22} = \frac{A^{22}}{\sin^2\alpha}. \tag{9}$$

$$A'^i{}_j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}A^k{}_l = U^i{}_kU^{-1}{}^l{}_jA^k{}_l, \tag{10}$$

$$A'^1{}_1 = U^1{}_1U^{-1}{}^1{}_1A^1{}_1 + U^1{}_1U^{-1}{}^2{}_1A^1{}_2 + U^1{}_2U^{-1}{}^1{}_1A^2{}_1 + U^1{}_2U^{-1}{}^2{}_1A^2{}_2 = A^1{}_1 - \text{cotan}\alpha A^2{}_1. \tag{11}$$

$$A'^1{}_2 = U^1{}_1U^{-1}{}^1{}_2A^1{}_1 + U^1{}_1U^{-1}{}^2{}_2A^1{}_2 + U^1{}_2U^{-1}{}^1{}_2A^2{}_1 + U^1{}_2U^{-1}{}^2{}_2A^2{}_2 = A^1{}_1\cos\alpha + A^1{}_2\sin\alpha - A^2{}_1\frac{\cos^2\alpha}{\sin\alpha} - A^2{}_2\cos\alpha. \tag{12}$$

$$A'^2{}_1 = U^2{}_1 U^{-1}{}^1{}_1 A^1{}_1 + U^2{}_1 U^{-1}{}^2{}_1 A^1{}_2 + U^2{}_2 U^{-1}{}^2{}_1 A^2{}_1 + U^2{}_2 U^{-1}{}^2{}_1 A^2{}_2 = \frac{A^2{}_1}{\sin\alpha}. \tag{13}$$

$$A'^2{}_2 = U^2{}_1 U^{-1}{}^1{}_2 A^1{}_1 + U^2{}_1 U^{-1}{}^2{}_2 A^1{}_2 + U^2{}_2 U^{-1}{}^1{}_2 A^2{}_1 + U^2{}_2 U^{-1}{}^2{}_2 A^2{}_2 = \text{cotan}\alpha A^2{}_1 + A^2{}_2. \tag{14}$$

$$A'^j_i = \frac{\partial x^k}{\partial x'^i}\frac{\partial x'^j}{\partial x^l} A_k{}^l = U^{-1}{}^k{}_i U^j{}_l A_k{}^l, \tag{15}$$

$$A'^1_1 = U^{-1}{}^1{}_1 U^1{}_1 A_1{}^1 + U^{-1}{}^1{}_1 U^1{}_2 A_1{}^2 + U^{-1}{}^2{}_1 U^1{}_1 A_2{}^1 + U^{-1}{}^2{}_1 U^1{}_2 A_2{}^2 = A_1{}^1 - A_1{}^2\text{cotan}\alpha. \tag{16}$$

$$A'^2_1 = U^{-1}{}^1{}_1 U^2{}_1 A_1{}^1 + U^{-1}{}^1{}_1 U^2{}_2 A_1{}^2 + U^{-1}{}^2{}_1 U^2{}_1 A_2{}^1 + U^{-1}{}^2{}_1 U^2{}_2 A_2{}^2 = \frac{A_1{}^2}{\sin\alpha}. \tag{17}$$

$$A'^1_2 = U^{-1}{}^1{}_2 U^1{}_1 A_1{}^1 + U^{-1}{}^1{}_2 U^1{}_2 A_1{}^2 + U^{-1}{}^2{}_2 U^1{}_1 A_2{}^1 + U^{-1}{}^2{}_2 U^1{}_2 A_2{}^2 = A_1{}^1\cos\alpha - A_1{}^2\frac{\cos^2\alpha}{\sin\alpha} + A_2{}^1\sin\alpha - A_2{}^2\cos\alpha. \tag{18}$$

$$A'^2_2 = U^{-1}{}^1{}_2 U^2{}_1 A_1{}^1 + U^{-1}{}^1{}_2 U^2{}_2 A_1{}^2 + U^{-1}{}^2{}_2 U^2{}_1 A_2{}^1 + U^{-1}{}^2{}_2 U^2{}_2 A_2{}^2 = A_1{}^2\tan\alpha + A_2{}^2. \tag{19}$$

$$A'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} A_{kl} = U^{-1}{}^k{}_i U^{-1}{}^l{}_j A_{kl}, \tag{20}$$

$$A'_{11} = U^{-1}{}^1{}_1 U^{-1}{}^1{}_1 A_{11} + U^{-1}{}^1{}_1 U^{-1}{}^2{}_1 A_{12} + U^{-1}{}^2{}_1 U^{-1}{}^1{}_1 A_{21} + U^{-1}{}^2{}_1 U^{-1}{}^2{}_1 A_{22} = A_{11} \tag{21}$$

$$A'_{12} = U^{-1}{}^1{}_1 U^{-1}{}^1{}_2 A_{11} + U^{-1}{}^1{}_1 U^{-1}{}^2{}_2 A_{12} + U^{-1}{}^2{}_1 U^{-1}{}^1{}_2 A_{21} + U^{-1}{}^2{}_1 U^{-1}{}^2{}_2 A_{22} = A_{11}\cos\alpha + A_{12}\sin\alpha \tag{22}$$

$$A'_{21} = U^{-1}{}^1{}_2 U^{-1}{}^1{}_1 A_{11} + U^{-1}{}^1{}_2 U^{-1}{}^2{}_1 A_{12} + U^{-1}{}^2{}_2 U^{-1}{}^1{}_1 A_{21} + U^{-1}{}^2{}_2 U^{-1}{}^2{}_1 A_{22} = A_{11}\cos\alpha + A_{21}\sin\alpha \tag{23}$$

$$A'_{22} = U^{-1}{}^1{}_2 U^{-1}{}^1{}_2 A_{11} + U^{-1}{}^1{}_2 U^{-1}{}^2{}_2 A_{12} + U^{-1}{}^2{}_2 U^{-1}{}^1{}_2 A_{21} + U^{-1}{}^2{}_2 U^{-1}{}^2{}_2 A_{22} = A_{11}\cos^2\alpha + (A_{12}+A_{21})\sin\alpha\cos\alpha + A_{22}\sin^2\alpha. \tag{24}$$

c) Now let's perform the similarity transformation:

$$\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} = \begin{pmatrix} 1 & -\text{cotan}\alpha \\ 0 & \frac{1}{\sin\alpha} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}. \tag{25}$$

Then,

$$\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} = \begin{pmatrix} A_{11} - A_{21}\text{cotan}\alpha & (A_{11} - A_{22})\cos\alpha + A_{12}\sin\alpha - A_{21}\frac{\cos^2\alpha}{\sin\alpha} \\ \frac{A_{21}}{\sin\alpha} & A_{21}\text{cotan}\alpha + A_{22} \end{pmatrix} \tag{26}$$

We see that it corresponds to the transformation of $A^i{}_j$.

d) Then in tensor notation the similarity transformation is given by

$$A'^i{}_j = U^i{}_k U^{-1}{}^l{}_j A^k{}_l = U^i{}_k A^k{}_l U^{-1}{}^l{}_j. \tag{27}$$