Homework #9

Problem 2:

We have to solve the equation

$$y'' - 2xy' + 2\alpha y = 0. \tag{1}$$

a) We will propose a solution of the form:

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}. \tag{2}$$

(b)
Plugging (2) in (1) we obtain:

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{(k+\lambda-2)} - 2\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)x^{(k+\lambda)} + 2\alpha \sum_{\lambda=0}^{\infty} a_\lambda x^{(k+\lambda)} = 0. \tag{3}$$

For $\lambda = 0$ the lowest power of x is x^{k-2}. This term vanishes if $a_0 k(k-1) = 0$. If $a_0 \neq 0$ it means that $k = 0$ or $k = 1$ are the only possible values k that will solve (1).
b) If $k = 0$ in order to cancel the coefficient of x^j in (3) we obtain:

$$a_{j+2}(j+2)(j+1) - 2a_j j + 2\alpha a_j = 0, \tag{4}$$

and thus

$$a_{j+2} = \frac{2(j-\alpha)a_j}{(j+2)(j+1)}. \tag{5}$$

Let's take $a_0 \neq 0$. Since a_1 is then arbitrary we will take $a_1 = 0$ which means, from (5), that all the odd coefficients vanish. From (5) we obtain:

$$a_2 = -\alpha a_0, \tag{6}$$

and

$$a_4 = \frac{(2-\alpha)}{6} a_2 = -\frac{\alpha(2-\alpha)}{6} a_0, \tag{7}$$

then,

$$y(x) = a_0[1 - \alpha x^2 - \frac{\alpha(2-\alpha)}{6} x^4 + ...]. \tag{8}$$

If $k = 1$ canceling the coefficients of x^j in (3) we obtain:

$$a_{j+2}(j+3)(j+2) - 2a_j(j+1) + 2\alpha a_j = 0, \tag{9}$$

and thus

$$a_{j+2} = \frac{2(j+1-\alpha)a_j}{(j+3)(j+2)}. \tag{10}$$

Then,

$$a_2 = \frac{(1-\alpha)}{3} a_0, \tag{11}$$

and

$$a_4 = \frac{(3-\alpha)}{10}a_2 = \frac{(1-\alpha)(3-\alpha)}{30}a_0, \tag{12}$$

then,

$$y(x) = a_0[x + \frac{(1-\alpha)}{3}x^3 + \frac{(1-\alpha)(3-\alpha)}{30}x^5 + ...]. \tag{13}$$

c) I will provide the values of $H_n(x)$ in terms of a_0 and I will give the value of a_0 that for each n I find comparing with the tabulated values of the Hermite polynomials:
If $\alpha = 0$ I find that $H_0(x) = a_0$, $a_0 = 1$.
If $\alpha = 1$ I find that $H_1(x) = a_0 x$, $a_0 = 1$.
If $\alpha = 2$ I find that $H_2(x) = (-2x^2 + 1)a_0$, $a_0 = -4$.
If $\alpha = 3$ I find that $H_3(x) = (-\frac{2}{3}x^3 + x)a_0$, $a_0 = -12$.
Notice that if we assume $a_0 = 0$ and $a_1 \neq 0$ in (b) we obtain the same results.