Homework #9

Problem 3 - 15.3.1:

We need to express the potential of the given array of charges in terms of Legendre polynomials. If we chose the z axis parallel to the direction in which the charges are alligned and we put the charge $-2q$ at the origin we see that the system has azimuthal symmetry. We see that, except over a spherical surface of radius a, i.e., where the charges q are located, Laplace equation is satisfied. So we can divide the space in two regions: region I for $r \leq a$ and region II for $r \geq a$; we will propose solutions to the Laplace equation in both regions and we are going to find the values of the coefficients using b.c. at $r = 0$, $r = a$ and $r = \infty$. Thus we propose:

$$\Phi^I(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(cos\theta) - \frac{2q}{4\pi\epsilon_0 r}, \tag{1}$$

where we have used that the potential diverges at $r = 0$ due to the charge $-2q$ at the origin. That should provide the only term with negative powers of r so, using the principle of superposition we added the potential produced by $-2q$ at the origin. The rest of the coefficients in the expansion should contain only positive powers of r.

In region II, since the potential has to vanish at $r \to \infty$, only negative powers of r will appear and we propose:

$$\Phi^{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(cos\theta). \tag{2}$$

The coefficients A_l and B_l will be determined by the b.c. at $r = a$. Since we have two sets of constants we need two b.c.'s which are the following:

a) The potential is continuous at $r = a$ then

$$\Phi^I(r = a,\theta) = \Phi^{II}(r = a,\theta), \tag{3}$$

Thus, multiplying both sides of (3) by $P_m(cos\theta)$ and integrating over $cos\theta$ in the interval [-1,1], we find relationships between the coefficients A_l and B_l. However, notice that the case $l = 0$ has to be considered separately due to the extra term in (1). Thus from (3) we obtain the following relationships:

i) If $l = 0$:

$$A_0 = \frac{B_0}{a} + \frac{q}{2\pi\epsilon_0 a}. \tag{4}$$

ii) If $l \neq 0$:

$$A_l = \frac{B_l}{a^{2l+1}}. \tag{5}$$

The second condition at $r = a$ is that there is a jump in the normal component of the electric field which is equal to $\sigma(\theta)/\epsilon_0$. Here σ is the surface density of charge which is due to the two charges q placed at $(r,\theta) = (a,0)$ and (a,π). We need to express σ as a function of θ remembering that the integral of the charge density over the total surface of the sphere has to give us the total charge which is $2q$. Following the steps we followed in class we see that

$$\sigma(\theta) = \frac{q}{2\pi a^2}[\delta(cos\theta - 1) + \delta(cos\theta + 1)]. \tag{6}$$

Since the component of the electric field normal to the surface where the b.c.'s are imposed is $E_n = E_r$, in terms of the potential it becomes $-\frac{\partial\Phi}{\partial r}$ and the condition is

$$\frac{\partial\Phi^{II}}{\partial r}|_{r=a} - \frac{\partial\Phi^{I}}{\partial r}|_{r=a} = -\frac{\sigma}{\epsilon_0}. \tag{7}$$

Then using (7) we obtain

$$\sum_l [-(l+1)\frac{B_l}{a^{l+2}} P_l(cos\theta) - l A_l a^{l-1} P_l(cos\theta) - \frac{q}{2\pi\epsilon_0 a^2} P_0(cos\theta)] = -\frac{q}{2\pi\epsilon_0 a^2}[\delta(cos\theta - 1) + \delta(cos\theta + 1)]. \tag{8}$$

Now we multiply both sides by $P_m(cos\theta)$ and integrate on $cos\theta$ between -1 and 1. We obtain, using the orthogonality properties of the Legendre polynomials,

$$-\frac{2(m+1)}{2m+1}\frac{B_m}{a^{m+2}} - \frac{2m}{2m+1}A_m a^{m-1} - \frac{2q}{2\pi\epsilon_0 a^2}\delta_{m,0} = -\frac{q}{2\pi\epsilon_0 a^2}(P_m(1) - P_m(-1)). \quad (9)$$

We know that $P_m(1) = 1$ for all m and $P_m(-1) = (-1)^m$. This means that (9) is 0 if m is odd and $-\frac{q}{\pi\epsilon_0 a^2}$ if m is even. Then if we replace in (9) A_m with the expressions in terms of B_m that we found before ((4) and (5)) we obtain that for m odd:

$$B_m = A_m = 0. \quad (10)$$

For $m = 0$

$$B_0 = 0,$$

and

$$A_0 = \frac{q}{2\pi\epsilon_0 a}. \quad (11)$$

For m even and different from 0:

$$B_m = \frac{qa^m}{2\pi\epsilon_0}$$

and

$$A_m = \frac{q}{2\pi\epsilon_0 a^{m+1}}. \quad (12)$$

Putting back the values of the coeficients in (1) and (2) we obtain:

$$\Phi^I(r,\theta) = \frac{q}{2\pi\epsilon_0 a}\sum_{j=0}^{\infty}\frac{r^{2j}}{a^{2j}}P_{2j}(cos\theta) - \frac{q}{2\pi\epsilon_0 r}, \quad (13)$$

and

$$\Phi^{II}(r,\theta) = \frac{q}{2\pi\epsilon_0 r}\sum_{j=1}^{\infty}\frac{a^{2j}}{r^{2j}}P_{2j}(cos\theta). \quad (14)$$

Notice that (14) is the expression valid far away from the array of charges and represents the potential of a quadrupole because the first term in the expansion goes like r^{-3}.