Homework #9

Problem 4 - 15.3.2:

We could solve this problem as we solved Problem 12.1.1 but with that technique we'd have to separate the space in 3 regions because now there is charge at $r = a$ and at $r = 2a$ and it will be very tedious to find the solution. However, in class we found that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(cos\theta), \tag{1}$$

where $r_>$ ($r_<$) indicates the larger (smaller) between r and r' and θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$..

Then, knowing that a charge q at $\mathbf{r}'$ produces a potential at $\mathbf{r}$ given by $\frac{q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$ and using (1), for the array of charges given and for $r \geq 2a$ we have:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} [-2\frac{a^l}{r^{l+1}} P_l(cos\theta) + 2\frac{(a)^l}{r^{l+1}} P_l(cos(\pi - \theta)) - \frac{(2a)^l}{r^{l+1}} P_l(cos(\pi - \theta)) + \frac{(2a)^l}{r^{l+1}} P_l(cos\theta)]. \tag{2}$$

Notice that $P_l(cos(\pi - \theta))) = P_l(-cos\theta) = (-1)^l P_l(cos\theta)$ then (2) becomes:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} [-2\frac{a^l}{r^{l+1}} + 2(-1)^l \frac{(a)^l}{r^{l+1}} - (-1)^l \frac{(2a)^l}{r^{l+1}} + \frac{(2a)^l}{r^{l+1}}] P_l(cos\theta). \tag{3}$$

We see that the terms with l even vanish and if $l = 1$ the coefficient is also 0. Thus, the expression is given by a sum over odd values of l with $l = 3$ being the first term:

$$\Phi(r,\theta) = \frac{q}{\pi\epsilon_0} \sum_{j=1}^{\infty} \frac{(2^{2j} - 1)a^{2j+1}}{r^{2j+2}} P_{2j+1}(cos\theta). \tag{4}$$

Notice that (4) is the expression valid far away from the array of charges and represents the potential of an octupole because the first term in the expansion goes like r^{-4}.