Homework #9

Problem 6 - 15.2.12:

Let's assume that the ring has charge q then it constitute a density of charge σ at $r = a$ given by

$$\sigma(\theta) = \frac{q}{2\pi a^2}\delta(cos\theta), \tag{1}$$

since $\int_S \sigma dS = q$. The problem has azimuthal symmetry and the Laplace equation is satisfied for $r < a$ and $r > a$. Thus we divide space in two regions proposing solutions of the Laplace equation in each of them and adjusting the coefficients using the b.c.'s at $r = 0$, $r = a$ and $r \to \infty$. For $r < a$, we propose

$$\Phi^I(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(cos\theta), \tag{2}$$

and for $r > a$ we propose:

$$\Phi^{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(cos\theta). \tag{3}$$

At $r = a$ the potential is continuous so

$$\Phi^I(r = a) = \Phi^{II}(r = a), \tag{4}$$

and the jump in the normal component of the electric field is proportional to σ which gives the condition:

$$\frac{\partial \Phi^{II}}{\partial r}|_{r=a} - \frac{\partial \Phi^I}{\partial r}|_{r=a} = -\frac{\sigma}{\epsilon_0}. \tag{5}$$

From (4) we obtain:

$$A_l = \frac{B_l}{a^{2l+1}}. \tag{6}$$

Eq.(5) gives

$$\sum_l [-(l+1)\frac{B_l}{a^{l+2}} P_l(cos\theta) - l A_l a^{l-1} P_l(cos\theta)] = -\frac{q}{2\pi\epsilon_0 a^2}\delta(cos\theta). \tag{7}$$

Using orthogonality of the P_l's and plugging (6) in (7) we obtain:

$$B_l = \frac{q}{4\pi\epsilon_0} a^l P_l(0). \tag{8}$$

Using (12.34) in the book and (6) we find that $A_l = B_l = 0$ if l is odd and

$$A_{2n} = \frac{B_{2n}}{a^{4n+1}} = \frac{q}{4\pi\epsilon_0 a^{2n+1}} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}. \tag{9}$$

Replacing (9) in (2) and (3) we obtain:

$$\Phi^I(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} a^{2n+1} (n!)^2} r^{2n} P_{2n}(cos\theta). \tag{10}$$

which is the result requested by the problem (the potential for $r < a$) and

$$\Phi^{II}(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} r^{2n+1} (n!)^2} a^{2n} P_{2n}(cos\theta). \tag{11}$$

(10) and (11) can be compactified in the expression:

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \frac{r_<^{2n}}{r_>^{2n+1}} P_{2n}(cos\theta). \tag{12}$$