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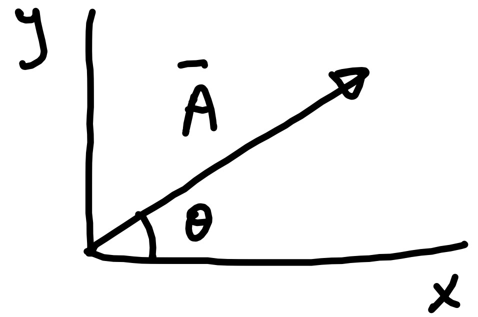
Review vectors in Ch. 1.7.

Vectors and Tensors

Vector:

K:



$|\bar{A}|$: magnitude

θ : directrou

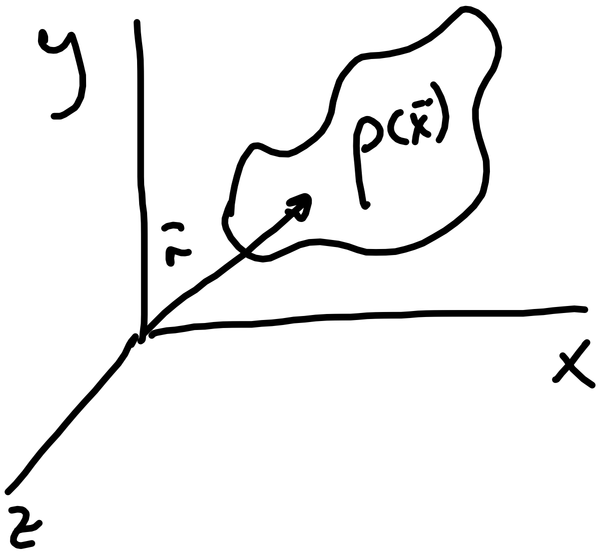
$|\bar{A}|$ is a scalar. It has the same value in any reference frame.

As a tensor a vector is a tensor of rank 1.

Examples of tensors:

rank	notation	name	Examples
0	a	scalar	speed, temperature time, energy.
1	a_i	vectors	velocity, position, force
2	a_{ij}	matrix	tensor of inertia quadrupole moment
3	a_{ijk}	cubic array	octupole moment
4	a_{ijkl}	hypercubic array	stress-tensor
⋮			

Multipole expansion defines tensors of all ranks:



$q_0 = \int \rho(\bar{x})\, dV$ charge (scalar)

$p_i = \int \rho(\bar{x})\, x_i \, dV$ dipole

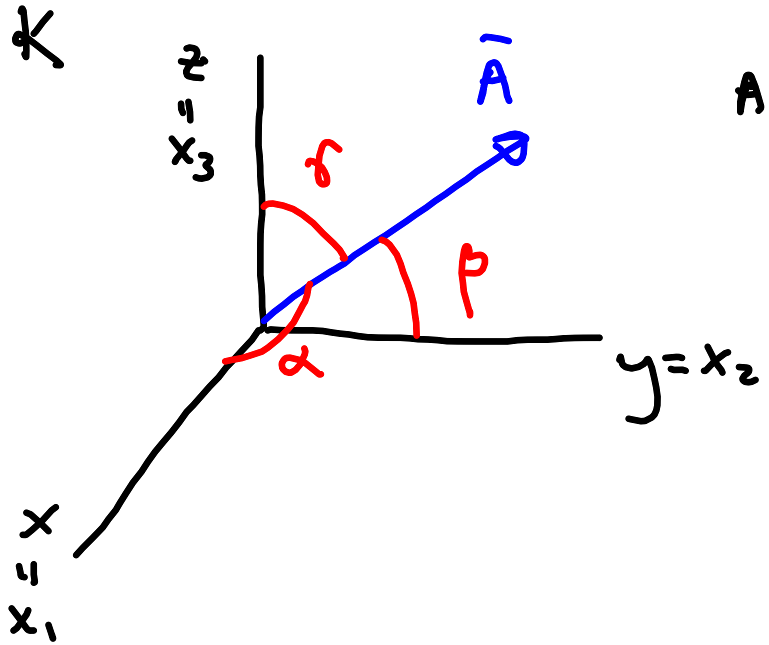
$q_{ij} = \int \rho(\bar{x})\, x_i x_j \, dV$ quadrupole

$q_{ijk} = \int \rho(\bar{x})\, x_i x_j x_k \, dV$ octupole

$\vdots$

Indices range from 1 to N where N is the space's dimension.

Tensors of rank 1:



magnitude: $|\bar{A}|$

$$A = |\bar{A}| = (A_x^2 + A_y^2 + A_z^2)^{1/2}$$

(HW #1)

$$A_x = A \underline{\cos\alpha}$$
$$A_y = A \underline{\cos\beta}$$
$$A_z = A \underline{\cos\gamma}$$

director cosines

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

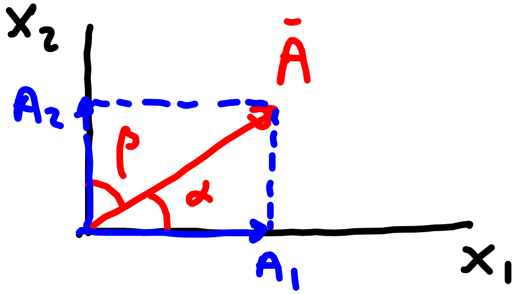
(HW#1).

$|\bar{A}| = 0$ if

Systems of Reference

$N=2$

Cartesian System:



$x_1 \perp x_2$

components from orthogonal projection

$\bar{A} = (A_1, A_2) = A_i = A^i$

components from parallel projection

$A_1 = A\cos\alpha$

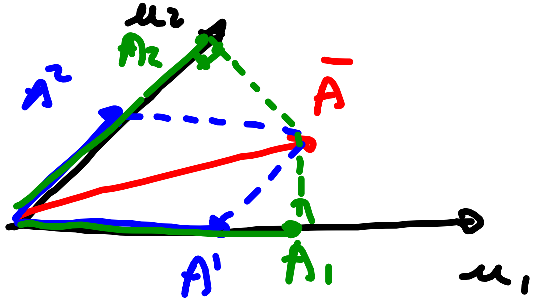
$A_2 = A\cos\beta = A\sin\alpha$

$A = |\bar{A}| = (A_1^2 + A_2^2)^{1/2}$

$\cos^2\alpha + \cos^2\beta = 1$

covariant and contravariant are equal.

Oblique system (common in condensed matter)



$\bar{A} = (A^1, A^2) \equiv A^i$ (parallel projection)

$\bar{A} = (A_1, A_2) \equiv A_i$ (perpendicular projection)

Two sets of possible components,

A_i : covariant
A^i : contravariant

} we'll see why later

General Definition of Vector

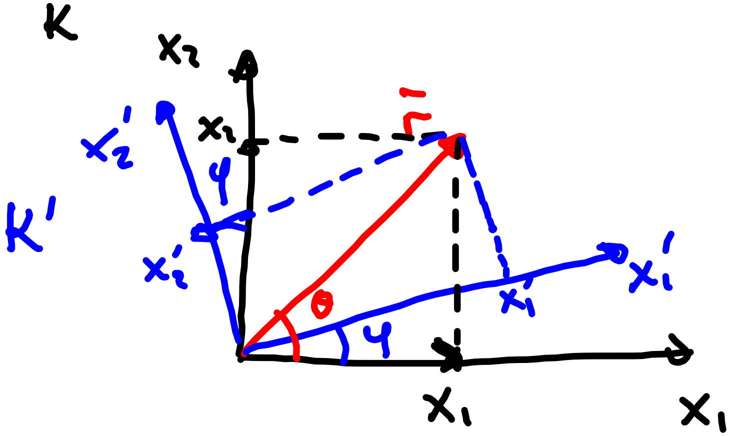
We select an special entity as our prototype vector. We study how its components transform from a reference frame K to a rotated reference frame K'. Then if we find anothe entity whose components transform in the same way we call it a vector.

Prototy vector: $r^i \equiv \bar{r}$ vector position.

- r^i is a contravariant vector (we'll see later why).
- In physics contravariant tensors have units proportional to length while covariant tensors have units of $[\text{length}]^{-1}$!
- Contravariant (covariant) tensors both have contravariant and covariant components.

Transformation roules:

Prototype vector $\bar{r}$ in $N=2$: $\bar{r} = (x_1, x_2)$



$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

$$\begin{cases} x_1' = x_1\cos\varphi + x_2\sin\varphi \\ x_2' = -x_1\sin\varphi + x_2\cos\varphi \end{cases}$$

$$x_1' = r\cos(\theta-\varphi) = \overbrace{r\cos\theta}^{x_1}\cos\varphi + \underbrace{r\sin\theta}_{x_2}\sin\varphi$$

$$x_2' = r\sin(\theta-\varphi) = r\sin\theta\cos\varphi - r\cos\theta\sin\varphi$$

The transformation is:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}}_{M} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

column vector caracterizes contravariant components.

$$x'_i = \sum_{j=1}^{2} M_{ij} x_j \equiv M_{ij} x_j$$

Einstein's notation
sum over repeated indices