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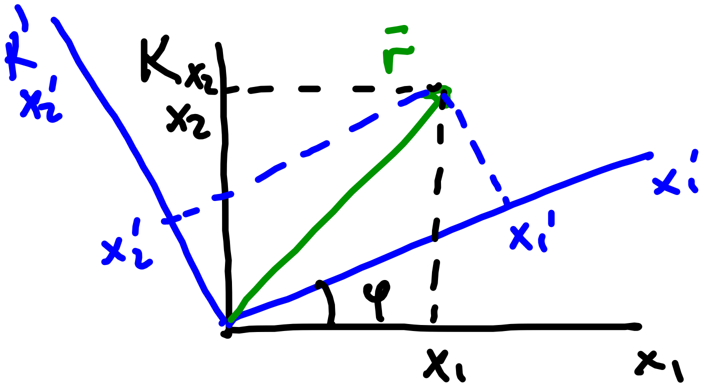
Last time:



we found that:

$$x_1' = x_1 \cos\varphi + x_2 \sin\varphi$$
$$x_2' = -x_1 \sin\varphi + x_2 \cos\varphi$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}}_{M} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$x_i' = \sum_j M_{ij} x_j$$

$$x_i' = M_{ij} x_j$$

sum over repeated indices

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} \\ \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} \end{pmatrix}$$

$$x'_i = M_{ij} x_j$$

or

$$\boxed{x'_i = \frac{\partial x'_i}{\partial x_j} x_j}$$

general transformation for the components of a contravariant vector such as $\bar{r}$.

In N dimensions the same transformation is valid with $i = 1, 2, \ldots N$.

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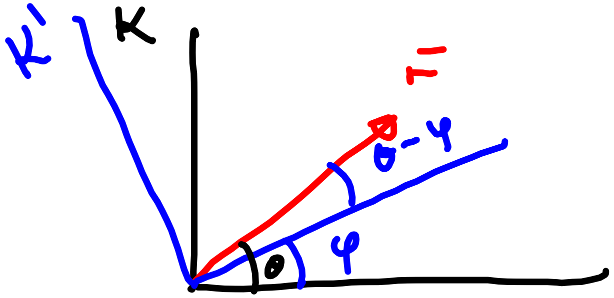
If $N=3$ in our example the rotation would be along the $z=x_3$ axis and we will add the relationship:

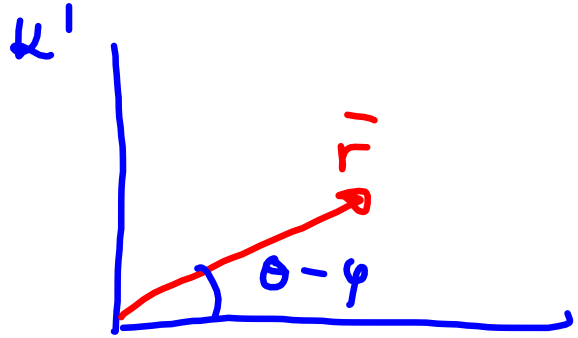
$$x'_3 = x_3 \qquad \text{but} \qquad \frac{\partial x'_3}{\partial x_j} = \begin{cases} 1 & \text{for } j=3 \\ 0 & \text{for } j=1,2 \end{cases}$$

$$M = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Why $\bar{r}$ is a contravariant vector?



Now we see that from K' perspective:



It seems as if $\bar{r}$ has rotated in the opposite direction than K.

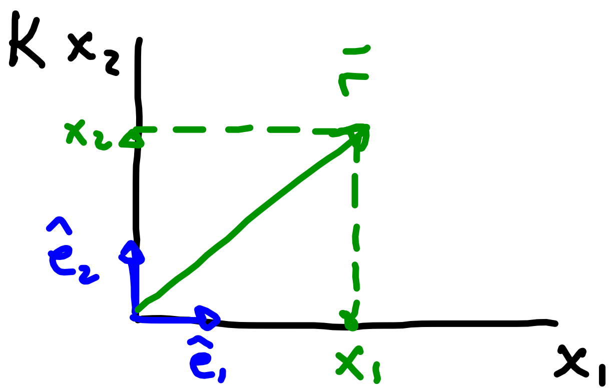
Notation:

$\bar{r} \equiv r^i$

contravariant.

Covariant vectors

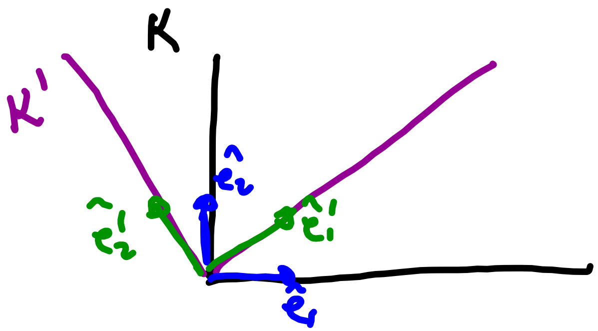
Canonical basis :



$$\bar{r} = x_1 \hat{e}_1 + x_2 \hat{e}_2 = \sum_{i=1}^{N} x_i \hat{e}_i$$

$$\equiv x^i \hat{e}_i$$

$$\hat{e}_i = \frac{\partial \bar{r}}{\partial x_i} = \frac{\partial}{\partial x_i}(x_1, x_2, x_3, \dots)$$

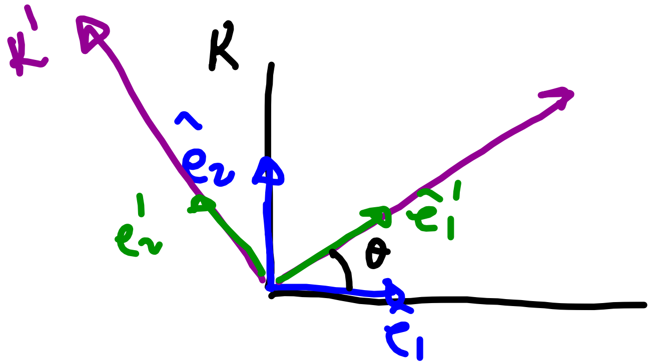


$\hat{e}_i$: rotate with the system and thus they are called <u>covariant</u>.

As a matrix we get that

$$\bar{r} = x^i \hat{e}_i = x^1 \hat{e}_1 + x^2 \hat{e}_2 = \underbrace{(\hat{e}_1, \hat{e}_2)}_{\text{row} \equiv \text{covariant}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\text{column} \equiv \text{contravariant.}}$$

Let's see how our covariant vectors transform:



$$\hat{e}_1' = \hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta$$

$$\hat{e}_2' = \hat{e}_2 \cos\theta - \hat{e}_1 \sin\theta$$

$$(\hat{e}_1', \hat{e}_2') = (\hat{e}_1, \hat{e}_2) \overbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{A}$$

Notice that

$AM = \mathbb{I}$ this means that $A = M^{-1}$

So the covariant vector transforms as the inverse of the contravariant one.

$$\begin{cases} x_1 = \cos x_1' - \sin x_2' \\ x_2 = \sin x_1' + \cos x_2' \end{cases}$$

You can see that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_1}{\partial x_2'} \\ \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_2}{\partial x_2'} \end{pmatrix}$$

$$\boxed{A^i{}_j = \frac{\partial x^i}{\partial x'^j}}$$

Covariant and contravariant transformations:

→ free index up

$$x^{i'} = \sum_j \frac{\partial x'^i}{\partial x^j} x^j \equiv \frac{\partial x'^i}{\partial x^j} x^j$$

$$\hat{e}'_j = \sum_i \hat{e}_i A^i{}_j = \sum_i \frac{\partial x^i}{\partial x'^j} \hat{e}_i \equiv \frac{\partial x^i}{\partial x'^j} \hat{e}_i$$

→ free index down

The index goes up for the contravariant components because the free index in the transformation is "up" and it goes down in the covariant transformation because the "free index" is down.

There are covariant and contravariant vectors but each vector can be written in terms of covariant and contravariant components.

Example: magnitude of the vector $\bar{r}$:

$$|\bar{r}|^2 = \bar{r}_i \cdot \bar{r}_i = \sum_{i=1}^{N} x_i^2$$

In terms of matrices we have to multiply the column vector that defines $\bar{r}$ by a row vector that defines the covariant components of $\bar{r}$:

$$|\bar{r}|^2 = (x_1, x_2, x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + x_2^2 + x_3^2$$

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In cartesian coordinates the contravariant and covariant coordinates of a vector are the same but this is not true in general.

Notice that

$$x'_i = \frac{\partial x^j}{\partial x'^i} x_j \qquad (x'_1, x'_2, x'_3) = (x_1, x_2, x_3) A$$

and

$$x'^i = \frac{\partial x'^i}{\partial x^j} x^j \qquad \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = M \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

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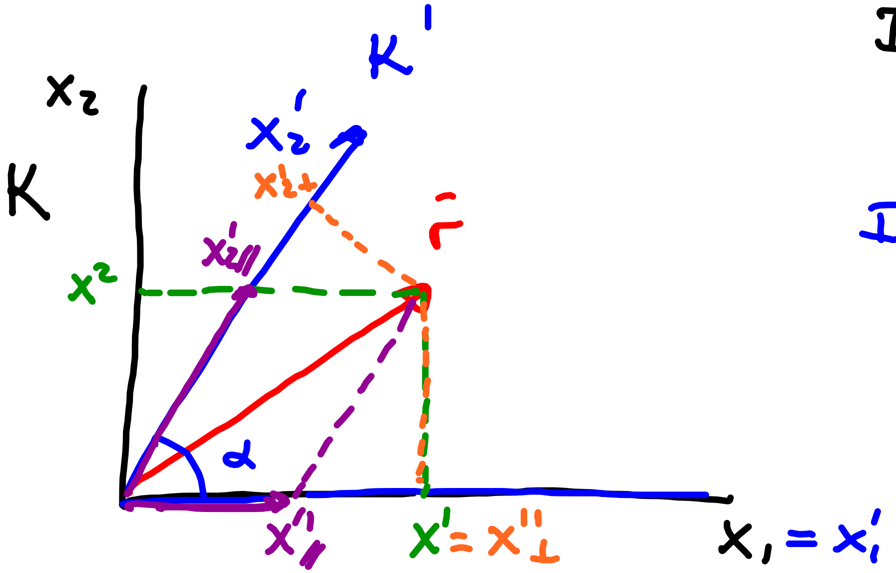
Non-trivial case: oblique axis.



In K:

$\bar{r} = x^j \hat{e}_j$

In K':

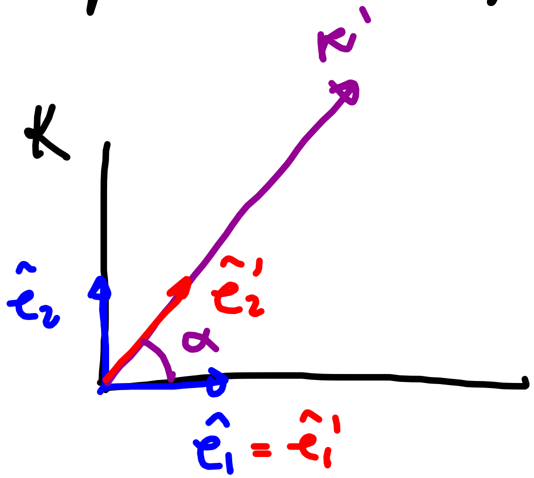
$\bar{r}' = \bar{r} = x'^j \hat{e}'_j$

what should I use? Parallel or perpendicular projections?

I want to obtain

$x'^i_{\parallel} = f(x^j)$ and $x'^i_{\perp} = g(x^j)$

Before let's find out how $\hat{e}_i$ transforms because this will tell us how the covariant components transform:



$$\hat{e}'_1 = \hat{e}_1$$

$$\hat{e}'_2 = \hat{e}_1 \cos\alpha + \hat{e}_2 \sin\alpha$$

$$(\hat{e}'_1, \hat{e}'_2) = (\hat{e}_1, \hat{e}_2) \underbrace{\begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}}_{A}$$

The components of $\bar{r}$ in K' that transforms like $\hat{e}'_i$ are going to be the covariant components.