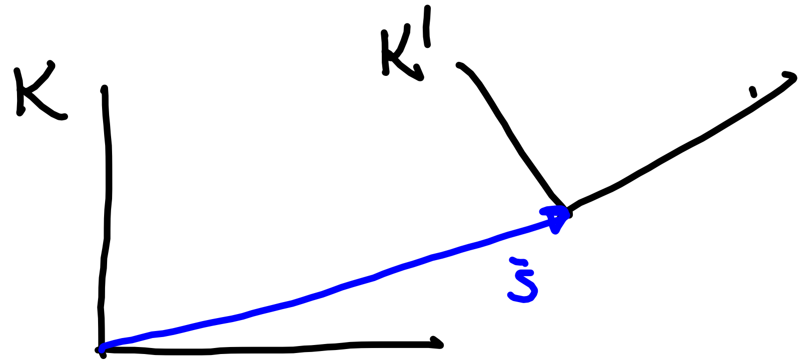
8/28

In reply to class question:

If K' is translated with respect to K

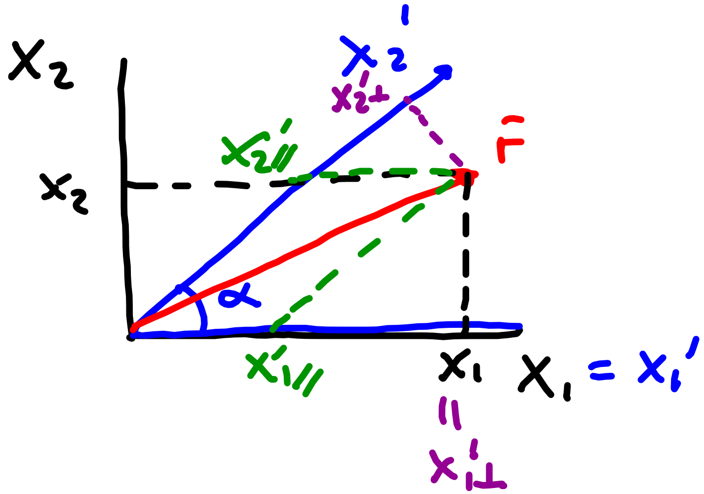


$$x'^i = M^i{}_j x^j + s'^i$$

rotation matrix.

Coordinates of $\bar{s}$ in K'

Oblique system



we found that

$\hat{e}_i$, covariant, transform

like:

$$(\hat{e}_1', \hat{e}_2') = (\hat{e}_1, \hat{e}_2) \underbrace{\begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}}_{A}$$

Covariant

transformation

Parallel Components

$$\begin{cases} x_1 = x'_{1\parallel} + x'_{2\parallel}\cos\alpha \\ x_2 = x'_{2\parallel}\sin\alpha \end{cases}$$

Then,

$$\begin{cases} x'_{1\parallel} = x_1 - x_2\cot\alpha \\ x'_{2\parallel} = \frac{1}{\sin\alpha}x_2 = \operatorname{cosec}\alpha \, x_2 \end{cases}$$

we see that

$$\begin{pmatrix} x'_{1\parallel} \\ x'_{2\parallel} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -\cot\alpha \\ 0 & \operatorname{cosec}\alpha \end{pmatrix}}_{M} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x'_{i\parallel} = x^{i'}$ contravariant components.

Perpendicular Components

$$x'_{1\perp} = x_1$$

$$x'_{2\perp} = x'_{2\parallel} + x'_{1\parallel}\cos\alpha =$$

$$= \frac{1}{\sin\alpha}x_2 + x_1\cos\alpha - x_2\frac{\cos^2\alpha}{\sin\alpha}$$

$$= x_1\cos\alpha + x_2\frac{\overbrace{(1-\cos^2\alpha)}^{\sin^2\alpha}}{\sin\alpha} =$$

$$= x_1\cos\alpha + x_2\sin\alpha$$

We see that

$$(x'_{1\perp}, x'_{2\perp}) = (x_1, x_2)\underbrace{\begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}}_{A}$$

$$= (x_1, x_2)A$$

$x'_{i\perp} = x'_i$ covariant components.

Check that

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j} = \frac{\partial x''^i}{\partial x^j} = \begin{pmatrix} 1 & -\cot\alpha \\ 0 & \csc\alpha \end{pmatrix}$$

$$A^i{}_j = \frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}$$

You can check that $A = M^{-1}$ or in tensor notation

$$A^i{}_j M^j{}_k = \delta^i{}_k \quad \begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i\neq k \end{cases}$$

Sum over j due to Einstein's notation

Not any array of N numbers is a vector.

A vector is an array whose components transform according to the rule:

$$x'^i = M^i{}_j\, x^j$$

or

$$x'_j = A^i{}_j\, x_i$$

Example: the electric field $\bar{E}$ is a vector in 3D. But in Minkowski space $\bar{E}$ does not transform as a vector.

(x, y, z, ct) 4-d space in which the components of $\bar{E}$ are part of a rank 2 tensor.

Vector operations:

- Scalar or dot product.

$$\bar{A} \cdot \bar{B} = \sum_{i=1}^{N} A_i B_i = \bar{B} \cdot \bar{A}$$

- In terms of matrices:

$$(A_1, A_2, \ldots A_N) \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{pmatrix} = A_i B^i \quad \text{summing over } i$$

also you can write

$$A^i B_i = S$$

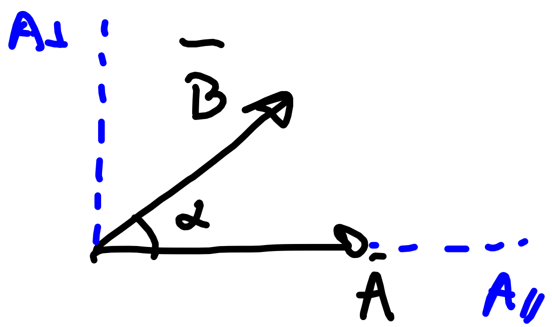
scalar - tensor of rank 0.

Norm or length of a vector:

$$\bar{A}\cdot\bar{A} = a_i a^i = \sum_{i=1}^{N} a_i^2 = |A|^2$$

square of the length. SCALAR.

$$|\bar{A}| = (\bar{A}\cdot\bar{A})^{1/2} \equiv A$$

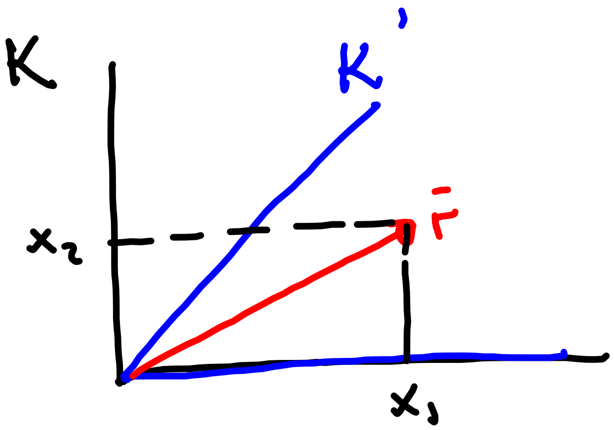


$$\bar{A}\cdot\bar{B} = \underbrace{A_\parallel}_{=A} \underbrace{B_\parallel}_{B\cos\alpha} + \overbrace{A_\perp}^{0 =} B_\perp =$$

$$= AB\cos\alpha$$

$$\cos\alpha = \frac{\bar{A}\cdot\bar{B}}{AB} = \frac{a_i b^i}{(a_j a^j)^{1/2}(b_k b^k)^{1/2}}$$ scalar.

Scalar product in the oblique system:



In K:

$r^2 = x_i x^i = x_1^2 + x_2^2$

In K':

$r^2 \equiv r'^2 = x'_i x'^i =$

$= x'_1 x'^1 + x'_2 x'^2 =$

writing $x'^i = f(x_i)$ and $x'_i = g(x_i)$ found in page 2.

$= x_1 (x_1 - \cot\alpha \, x_2) + (x_1 \cos\alpha + x_2 \sin\alpha) \dfrac{x_2}{\sin\alpha} =$

$= x_1^2 \cancel{- x_1 x_2 \cot\alpha} \cancel{+ x_1 x_2 \cot\alpha} + x_2^2 =$

$= x_1^2 + x_2^2 = r^2$

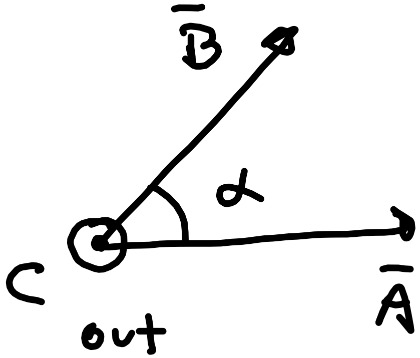
Check at home that $x'^i x'^i \neq r$

and $x'_i x'_i \neq r$.

Cross or vector product:

$$\bar{C} = \bar{A} \times \bar{B} = AB \sin\alpha$$

Direction obtaind using right hand roule.



Another way is using the determinat form:

$$\bar{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - B_x A_y)$$

Clearly:

$$\bar{B} \times \bar{A} = -\bar{A} \times \bar{B}$$

In tensor notation:

$$C_i = \varepsilon_{ijk} A^j B^k$$

ε_{ijk} : Levi-Civita tensor

$$\varepsilon_{ijk} \begin{cases} 0 & \text{if any of the indices are the same.} \\ 1 & \text{if all the indices are different and in cyclic order} \\ -1 & \text{if all the indices are different but not in cyclic order.} \end{cases}$$

$\varepsilon_{112} = 0$ $\varepsilon_{123} = 1$ $\varepsilon_{213} = -1$

Example:

$$C_i = \varepsilon_{ijk} A^j B^k$$

Then $C_x = C_1$

$$C_1 = \overbrace{\varepsilon_{123}}^{1} A^2 B^3 + \overbrace{\varepsilon_{132}}^{-1} A^3 B^2 =$$

only non-zero terms

$$= A^2 B^3 - A^3 B^2$$

then

$$C_x = A_y B_z - A_z B_y$$

Vectors and covectors in real, reciprocal and dual space.

Solid State Physics:

Crystals: are periodic arrays of ions in real space.

The ions are attached to a Bravais lattice.