12/2

Quantum mechanical momentum representation:

$\psi(x)$ wave function in real space

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, x\, \psi(x)\, dx$$

We want to find $\langle p \rangle$ the average value of the momentum and the function $\tilde{\psi}(p)$ that will allow us to obtain $\langle p \rangle$.

$$\langle p \rangle = \int_{-\infty}^{\infty} \tilde{\psi}^*(p)\; p\; \tilde{\psi}(p)\, dp$$

find p and $\tilde{\psi}(p)$.

$|\tilde{\psi}(p)|^2 dp$ = probability of finding the particle with momentum in the $p - p + dp$ interval

and

$$\int_{-\infty}^{\infty} |\tilde{\psi}(p)|^2 dp = 1$$

Let's find $FT\,\psi(x) =$

$$FT\,\psi(x) = \tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ikx}\, dx \qquad (*)$$

Planck's constant

$\lambda = \frac{h}{p}$ (de Broglie assumption)

momentum

wavelength

$$k = \frac{2\pi}{\lambda} \Rightarrow \frac{2\pi}{k} = \frac{h}{p} \text{ then } p = \frac{h}{2\pi} k = \hbar k \qquad (**)$$

Combining $(*)$ and $(**)$:

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)\, e^{-\frac{ipx}{\hbar}}\, dx$$

also

$$\tilde{\psi}^*(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi^*(x)\, e^{\frac{ipx}{\hbar}}\, dx \, .$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \tilde{\psi}^*(p)\; p \;\psi(p)\; dp =$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\hbar} \iint \psi^*(x')\, e^{\frac{ipx'}{\hbar}}\, p \;\psi(x)\, e^{-ipx/\hbar}\, dp\, dx\, dx'$$

(1)

Let's calculate

$$\int_{-\infty}^{\infty} p \, e^{ip\frac{(x'-x)}{\hbar}} \, dp = -\frac{2\pi\hbar^2}{i}\delta'(x-x') =$$

$$= 2\pi i \hbar^2 \delta(x-x')$$

②

We know that

$$\hbar \int_{-\infty}^{\infty} e^{ik(x'-x)} \, dk = 2\pi\hbar \, \delta(x-x')$$

then

$$\frac{d}{dx}\int_{-\infty}^{\infty} e^{ip\frac{(x'-x)}{\hbar}} dp = 2\pi\hbar \, \delta'(x-x')$$

$$-\frac{i}{\hbar}\int_{-\infty}^{\infty} p \, e^{ip\frac{(x'-x)}{\hbar}} dp$$

Plugging ② in ① we obtain:

$$\langle p \rangle = \frac{\cancel{2\pi} i \hbar^2}{\cancel{2\pi \hbar}} \iint \psi^*(x') \, \psi(x) \, \delta'(x - x') \, dx \, dx' =$$

We know that

$$\int f(x) \, \delta(x - x') \, dx = -f'(x')$$

$$= -i\hbar \int \psi^*(x') \frac{d\psi}{dx'} dx'$$

Then $\boxed{p = -i\hbar \frac{d}{dx}}$

Notice that

$$\int |\tilde{\psi}(p)|^2 \, dp = 1$$ since

$$\int |\psi(x)|^2 \, dx = 1$$ due to Perceval relationship.

Grades:

Hw : x /130.

MT's: x /140.

Participation : 10/10 (when you bring or email class evaluation form).

Final : x /120.

↳ But there are Bonus points.