11/11

Tips for the test:

- Bring a calculator (no phone).
- Bring book. (tables).
- Provide numerical values when asked.
- Remember:

$$x_i = g_{ij} x^j \qquad F^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\rho} F_{\gamma\rho}$$

$$x^i = g^{ij} x_j$$

In Minkowsky space:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$\mu = 0, 1, 2, 3$

$x_0 = ct$
$x_1 = x$
$x_2 = y$
$x_3 = z$

$$A_\mu A^\mu = g_{\mu\nu} A^\nu A^\mu = (A^0)^2 - |\bar{A}|^2$$

$$\partial_\lambda = (\partial_0, \bar{\nabla})$$

$$\partial^\lambda = g^{\lambda\mu} \partial_\mu$$

$\bar{\nabla} r = ?$ Calculate this using tensor notation.

$\partial_i (x_j x^j)^{1/2} = B_i$ covariant components of vector $\bar{B} = \bar{\nabla} r$

$$\partial_i (x_j x^j)^{1/2} = \frac{1}{2} \frac{(\partial_i x_j\, x^j + x_j \partial_i x^j)}{\underbrace{(x_j x^j)^{1/2}}_{r}}$$

$\partial_i \equiv \frac{\partial}{\partial x^i}$

$x_j = g_{jk} x^k$

$$= \frac{1}{2r} \left(\partial_i g_{jk} x^k\, x^j + x_j \delta_i^{\ j} \right) =$$

$$= \frac{1}{2r} \left(g_{jk} \delta_i^{\ k} x^j + x_i \right) =$$

$$= \frac{1}{2r} \left(g_{ji} x^j + x_i \right) = \frac{1}{2r} (x_i + x_i) = \frac{2x_i}{2r} = \frac{x_i}{r}$$

Then in vector notation

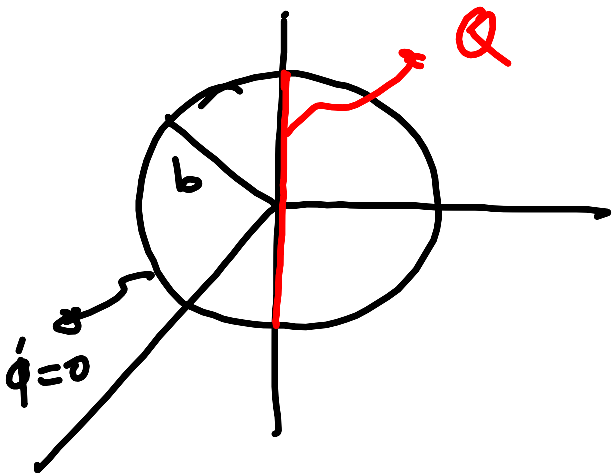
$$\bar{\nabla} r = \frac{\bar{x}}{r}$$

- $\bar{\nabla} \times \bar{B} \longrightarrow \varepsilon^{ijk} \partial_j B_k$

- Separation of variables:
  - Cartesian
  - spherical $\left\{ \begin{array}{l} \text{azimuthal symmetry} \\ \text{no azimuthal symmetry} \end{array} \right.$

- Theorem of superposition.
- Frobenius method.

Green Functions Examples.



we cannot use separation of variables since $\nabla^2\Phi \neq 0$ on the z axis. But we can use Green functions.

$$\rho(\bar{r}') = \frac{Q}{2b}\,\frac{1}{2\pi r'^2}\left[\delta(\cos\theta'-1) + \delta(\cos\theta'+1)\right]$$

$$\int_V \rho(\bar{r}')\, d^3r' = Q$$

$$\Phi(\bar r) = \frac{1}{4\pi\varepsilon_0}\int_V G(\bar r,\bar r')\,\rho(\bar r')\,d^3r' =$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^b r'^2\,dr'\int_{-1}^{1} d(\cos\theta')\int_0^{2\pi} d\varphi'\;\rho(\bar r')\,G(\bar r,\bar r') =$$

$$= \frac{Q}{2\pi\, 2b}\,\frac{4\pi}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\,Y_{\ell m}(\theta,\varphi)$$

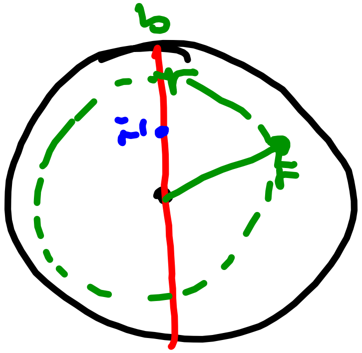
$$\int_0^b\left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^{\ell}r'^{\ell}}{b^{2\ell+1}}\right] dr'\int_{-1}^{1} d(\cos\theta')\left[\delta(\cos\theta'+1) + \right.$$

$$\left. + \,\delta(\cos\theta'-1)\right]\underbrace{\int_0^{2\pi} d\varphi'\; Y_\ell^{m}(\theta',\varphi')}_{2\pi\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta')\,\delta_{m,0}}$$

$$= \frac{Q}{4\pi 2b\varepsilon_0} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \int_0^b \left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}} \right] dr'$$

$$\underbrace{\left[P_\ell(1) + P_\ell(-1) \right]}_{\substack{2 \text{ for } \ell \text{ even} \\ 0 \text{ for } \ell \text{ odd}}} =$$

$$= \frac{Q}{4\pi b\varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \underbrace{\int_0^b \left[\frac{r_<^{2j}}{r_>^{2j+1}} - \frac{r^{2j} r'^{2j}}{b^{4j+1}} \right] dr'}_{\int_0^r \frac{r'^{2j}}{r^{2j+1}} dr' + \int_r^b \frac{r^{2j}}{r'^{2j+1}} dr' - \frac{r^{2j}}{b^{4j+1}} \int_0^b r'^{2j} dr'} =$$



$$= \frac{Q}{4\pi b \varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta)\Big[\underbrace{\int_0^r \frac{r'^{2j}}{r^{2j+1}}\,dr'}_{\frac{1}{r^{2j+1}}\frac{1}{2j+1} r'^{2j+1}\Big|_0^r} +$$

$$\frac{1}{r^{2j+1}}\ \frac{1}{2j+1}\ r'^{2j+1}\Big|_0^r$$

$$\underbrace{}_{\frac{1}{2j+1}}$$

$$+ \underbrace{\int_r^b \frac{r^{2j}}{r'^{2j+1}}\,dr'}_{} - \underbrace{\frac{r^{2j}}{b^{4j+1}} \int_0^b r'^{2j}\,dr'}_{}\Big]$$

if $j=0$ $\int_r^b \frac{1}{r'}dr' = \ln b/r$

$= -\ln r/b$

if $j \neq 0$

$$-\frac{1}{2j+2}\frac{r^{2j}}{r'^{2j+2}}\Big|_r^b =$$

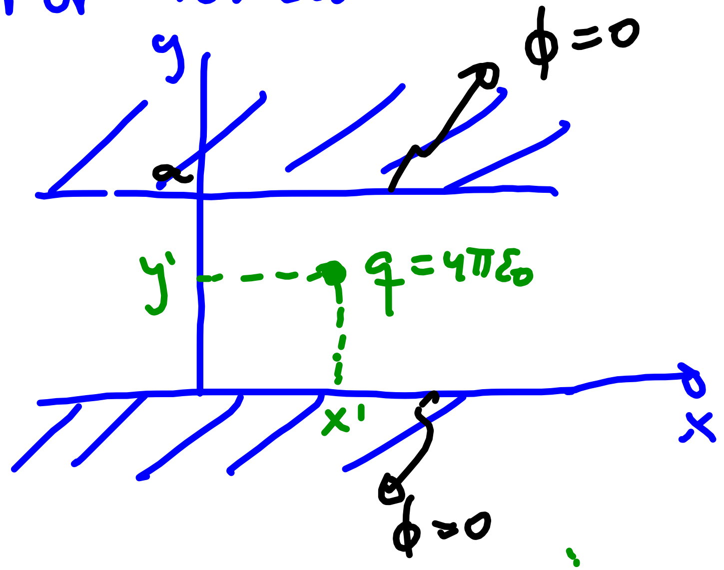
$$= -\frac{r^{2j}}{2j\,b^{2j}} + \frac{1}{2j}$$

$$\frac{1}{2j+1}\ r'^{2j+1}\Big|_0^b$$

$$\underbrace{\frac{r^{2j}}{b^{4j+1}(2j+1)} \qquad b^{2j+1}}_{\frac{-r^{2j}}{(2j+1)\,b^{2j}}}$$

$$= \frac{\alpha}{4\pi b \varepsilon_0} \Big\{ \Big[\sum_{j=1}^{\infty} P_{2j}(\cos\theta) \Big[\frac{1}{(2j+1)} - \frac{r^{2j}}{2j\, b^{2j}} + \frac{1}{2j} - \frac{r^{2j}}{b^{2j}} \frac{1}{2j+1} \Big] + \ln b/r$$

For homework:



Find $G(\bar{r}, \bar{r}')$ for this geometry.

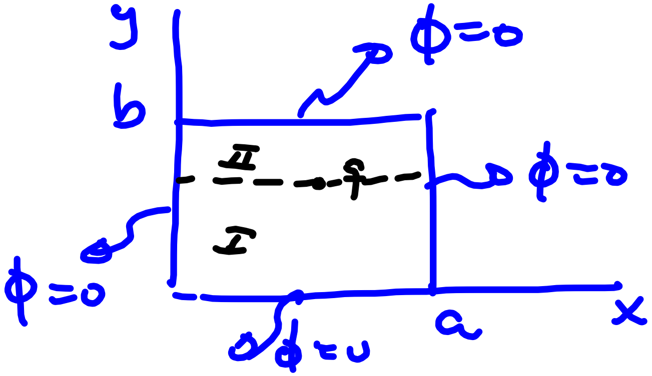
Solve using separation of variables finding $\phi(\bar{r})$ for a charge $q = 4\pi\varepsilon_0$ placed at $\bar{r}'$.

You should find that

$$G(x, x', y, y') = 4 \sum_{m=1}^{\infty} \frac{e^{\frac{m\pi}{a}(x_< - x_>)}}{m} \sin\frac{m\pi y}{a} \sin\frac{m\pi y'}{a}$$

where $x_<$ ($x_>$) is the smaller (larger) between x and x'.

Another problem:

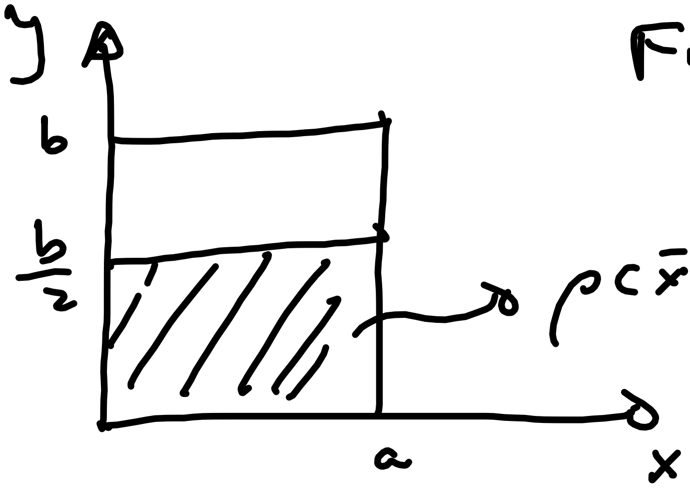


Find $G(x, x', y, y')$ with Dirichlet b.c.

Notice that you can put the singularity along x or along y depending on the problem you need to solve.

Problem #3 of HW #11:

Find $\phi(x,y)$ inside the box:



$$\rho(\bar{x}) = \sigma_0 \, \underbrace{\Theta\left(y-\frac{b}{2}\right)}_{\substack{1 \text{ if } y \le \frac{b}{2} \\ 0 \text{ otherwise.}}}$$

$$\phi(x,y) = \frac{1}{4\pi\varepsilon_0} \int_0^a dx' \int_0^b dy' \, G(x,x',y,y') \, \rho(x',y') =$$

$$= \frac{\sigma_0}{4\pi\varepsilon_0} \int_0^a dx' \int_0^{b/2} dy' \, G(x,x',y,y')$$

Notice that for $y > \frac{b}{2}$ in G we that $y_> = y$
and $y_< = y'$

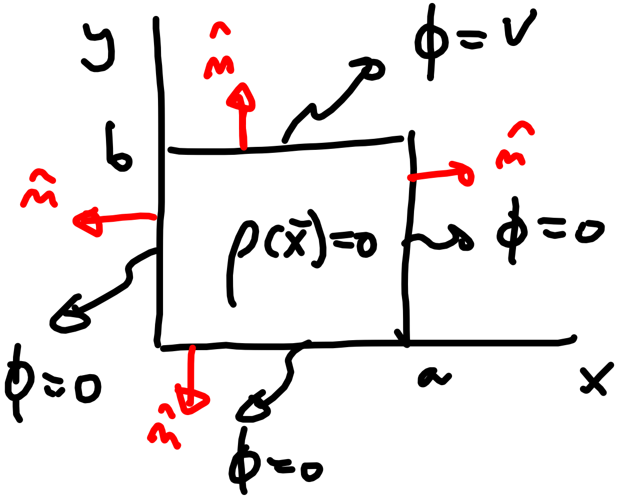
For $y < \frac{b}{2}$

$$\int_0^{b/2} dy'\, G(x, x', y, y') = \int_0^{y} dy'\, G + \int_y^{b/2} dy'\, G$$

here $y_< = y'$
$y_> = y$

here
$y_< = y$
$y_> = y'$

Other problems that you can solve with G:



We did it with separation of variables but it can be done with G.

$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon}\int_V G(\bar{x},\bar{x}')\overbrace{\rho(\bar{x}')}^{0}\,d^3x' - \frac{1}{4\pi}\oint_S ds'\,\phi_s \frac{\partial G}{\partial n'}\Big|_S$$

$$G(x, x', y, y') = 8 \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a}$$

$$\frac{\sinh\left[(b-y')\frac{n\pi}{a}\right] \sinh\frac{n\pi y}{a}}{\sinh\frac{bn\pi}{a}}$$

Since at the surface $y' = b$ and $y < b$ then $y_> = y'$ and $y_< = y$.

$$\left.\frac{\partial G}{\partial n'}\right|_s = \left.\frac{\partial G}{\partial y'}\right|_{y'=b}$$