11/18

Integral Transforms (Ch. 20).

$$\underbrace{g(\alpha)}_{\text{integral transform of } f} = \int_a^b f(t)\, \underbrace{K(\alpha, t)}_{\text{Kernel}}\, dt$$

function to be transformed.

The transform provides a mapping of a function $f(t)$ into another function $g(\alpha)$.

α and t are called conjugate variables.

Examples: x and k or ω and t.

Fourier transform

Kernel: $K(\alpha, t) = e^{-i\alpha t}$

$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\, dt \qquad (1)$$

$g(\alpha)$ provides the spectral decomposition of $f(t)$. If $\alpha \equiv \omega$ then $g(\alpha) = g(\omega)$ provides the decomposition in plane waves of $f(t)$.

$\{e^{-i\omega t}\}$ form a complete and orthogonal set of functions in $(-\infty, \infty)$.

Then:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\,dt = \delta(\omega-\omega')$$ orthogonality

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\,d\omega = \delta(t-t')$$ completitude

Antitransform:

Multiply ① by $\dfrac{e^{-i\omega t'}}{\sqrt{2\pi}}$ (assume $\alpha \equiv \omega$)

and integrate between $(-\infty,\infty)$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t'}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t'}\, d\omega \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, dt \underbrace{\int_{-\infty}^{\infty} e^{-i\omega(t'-t)}\, d\omega}_{2\pi\,\delta(t-t')} =$$

$$= \int_{-\infty}^{\infty} f(t)\, dt\, \delta(t-t') = f(t')$$

Now redefine $t' \to t$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t}\, d\omega$$

Generalization to 3D:

$$g(\bar{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\bar{r})\, e^{i\bar{k}\cdot\bar{r}}\, d^3r$$

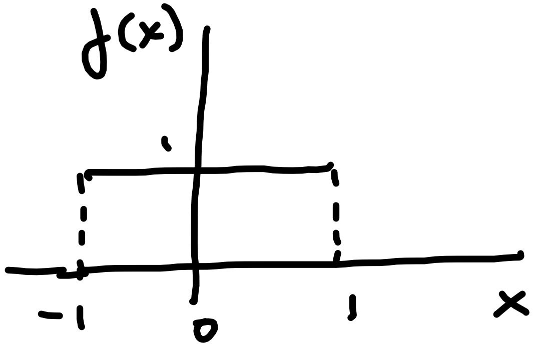
How you choose your system of coordinates may help simplify calculations. In some cases $\bar{k}\cdot\bar{r} = kr\cos\alpha$ you may select your axes so that $\alpha = \theta$. 

or $\hat{z} \parallel \bar{k}$.

The antitransform is given by:

$$f(\bar{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\bar{k})\, e^{-i\bar{k}\cdot\bar{r}}\, d^3k.$$

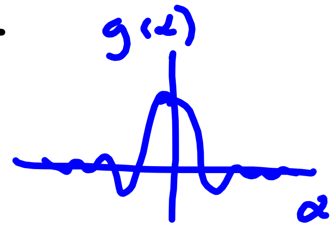
Examples:



$$f(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{i\alpha x}}{i\alpha}\Bigg|_{-1}^{1} =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(e^{i\alpha} - e^{-i\alpha})}{i\alpha} = \frac{1}{\sqrt{2\pi}} \frac{2\sin\alpha}{\alpha}$$



Now let's do the antitransform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\alpha)\, e^{-i\alpha x}\, d\alpha =$$

$$= \frac{2}{\left(\sqrt{2\pi}\right)^2} \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\, e^{-i\alpha x}\, d\alpha =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha} \left(\cos\alpha x - i \sin\alpha x\right) d\alpha =$$

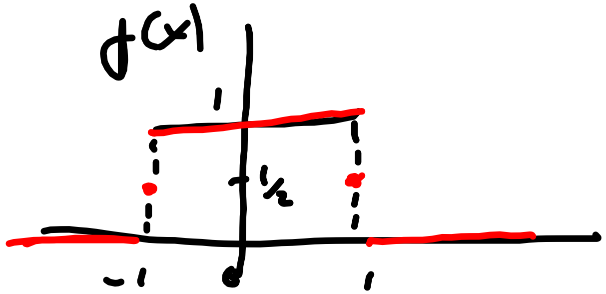
$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin\alpha \cos\alpha x}{\alpha}\, d\alpha \qquad (2)$$

Notice that if $x = \pm 1$ then (E) becomes

$$f(\pm 1) = \frac{2}{\pi} \int_0^\infty \frac{\sin\alpha \cos\alpha}{\alpha}\, d\alpha = \frac{1}{2}$$

At a discontinuity the antitransform gives you the average value of the function.

From $g(\alpha)$ we obtain:



Notice that using this we find that:

$$\underbrace{\int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha}\, d\alpha}_{\frac{\pi}{2} f(x)} = \begin{cases} \frac{\pi}{2} & \text{if } |x| < 1 \\ \frac{\pi}{4} & \text{if } |x| = 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Cosine and Sine Fourier Transforms.

Since $e^{ikx} = \cos kx + i \sin kx$

if we expand a function with even or odd symmetry only one of the 2 terms survives:

Cosine transform: (for even functions)

if $f_c(x) = f_c(-x)$ then

$$g_c(k) = \frac{1}{\sqrt{2\pi}} \, 2\int_0^\infty f_c(x) \cos kx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(x) \cos kx \, dx$$

and

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(k) \cos kx \, dk.$$

Notice that $g_c(k)$ is also even.

Sine transform: (odd functions)

$$f_s(x) = -f_s(-x) \quad \text{(odd)}.$$

$$g_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin kx \, dx$$

and

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(k) \sin kx \, dk$$

$g_s(k)$ is also odd.

Application to problems:

1) transform the problem.

2) Solve it

3) antitransform to obtain the answer.

Consider the Fourier transform of $f'(x) = \frac{df}{dx}$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$g_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{-ikx}\, dx \qquad (3)$$

Assume that $f(x)$ is a well behaved function so that $\lim_{|x|\to\infty} f(x) = 0$

Then we are going to integrate ③ by parts:

$$\int_a^b u'v\,dx = uv\Big|_a^b - \int_a^b u\,v'\,dx$$

$$g_1(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\,e^{-ikx}\,dx = \overbrace{\cancel{\frac{1}{\sqrt{2\pi}} f(x)\,e^{-ikx}\Big|_{-\infty}^{\infty}}}^{0}$$

$$- \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}\,dx = -ik\,g(k)$$

In general

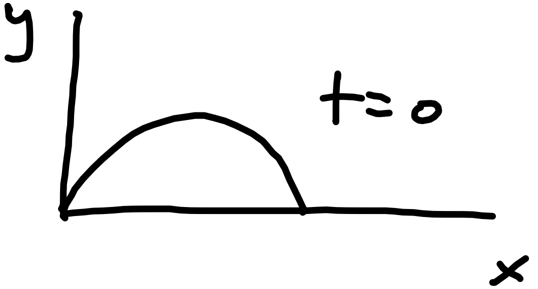
$$g_n(k) = (-ik)^n g(k)$$

where $g_n(k) = FT\left(\frac{d^n f(x)}{dx^n}\right)$.

A derivative of order n in x-space becomes a multiplication in k-space.

Now we can solve in a simpler way some differential equations:

Example:



$y(x,0) = f(x)$

$y(x,t) = ?$

$\left.\begin{array}{l} y(t=0) \\ y'(t=0) \end{array}\right\}$ initial conditions

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad (1)$$

Let's work in k- space:

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{y(k,t)}_{\text{indep of } x} \underbrace{e^{-ikx}}_{\text{all the } x \text{ dependence is here!}} dk \quad (2)$$

independent of t

Plug (2) in (1):

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -k^2 y(k,t)\, e^{-ikx} dk = \frac{1}{v^2} \int_{-\infty}^{\infty} \frac{\partial^2 y(k,t)}{\partial t^2} e^{-ikx} dk \quad (3)$$

Since (3) has to be valid for any arbitrary value of x:

$$-k^2 y(k,t) = \frac{1}{v^2} \frac{\partial^2 y(k,t)}{\partial t^2}$$

we see that

$y(k,t) = y_k e^{\pm ivkt}$

is solution.

We found that

$$y(k,t) = \underbrace{y_k}_{\text{independent of } t} e^{\pm i v k t} \qquad (4)$$

We also know that for $t=0$ $y(x,0) = f(x)$

Then

$$y_k = FT(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y_k e^{-ikx} dk \qquad (5)$$

Now once I got $y(k,t)$ I can obtain $y(x,t)$ by anti-Fourier transforming $y(k,t)$:

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(k,t) e^{-ikx} \, dk \stackrel{4}{=}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y_k \, e^{\pm i v k t} \, e^{-ikx} \, dk =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y_k \, e^{-ik(x \pm vt)} \, dk \stackrel{5}{=} f(x \pm vt)$$

The actual solution to the problem will be

$$y(x,t) = A\,f(x+vt) + B\,f(x-vt)$$

with A and B determined by

$y(x,0)$ and $y'(x,0)$ [initial conditions].