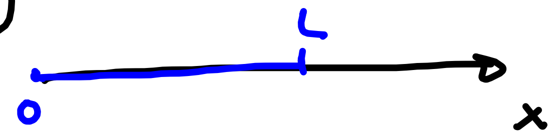
11/21

Heat flow equation in 1D.

$$\underbrace{\frac{\partial \psi(x,t)}{\partial t}}_{\text{change in heat stored}} = a^2 \underbrace{\frac{\partial^2 \psi(x,t)}{\partial x^2}}_{\text{net heat out}} \qquad (1)$$



$T(x,t)$

$$-\frac{dQ}{dt} = -c\rho\frac{dT}{dt}$$

$$\vec{\nabla} Q = \vec{\nabla}\cdot(-\sigma\vec{\nabla} T) = -\sigma \nabla^2 T$$

To find $\psi(x,t)$ we will work in Fourier space.

Let's write $\psi(x,t)$ in terms of its FT $\psi(k,t)$:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k,t)\, e^{-ikx}\, dk \qquad \text{(2)}$$

Plugging (2) in (1):

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \psi(k,t)}{\partial t}\, e^{-ikx}\, dk = \frac{a^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -k^2 \psi(k,t)\, e^{-ikx}\, dk$$

For each value of k we have to find $\psi(k,t)$ that satisfies:

$$\frac{\partial \psi(k,t)}{\partial t} = -a^2 k^2 \psi(k,t)$$

$$\int_{\psi(t=0)}^{\psi(t)} \frac{d\psi}{\psi} = -a^2 k^2 \int_0^t dt'$$

$$\ln \psi(k,t) - \ln \psi(k,0) = -a^2 k^2 t \qquad (3)$$

Exponentiating both sides of 3 :

$$\psi(k,t) = \psi(k,0)\, e^{-k^2a^2t} \quad (4)$$

Plugging 4 in 2 :

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\psi(k,0)\, e^{-k^2a^2t}}_{\psi(k,t)}\, e^{-ikx}\, dk$$

The solution depends on the initial condition.

If $\psi(k,0) = c$ then

$$\psi(x,t) = \frac{c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 a^2 t - ikx}\, dk =$$

$$= \frac{c}{a\sqrt{2t}}\, e^{-x^2/4a^2 t}$$

table for FT.

Green function for the inhomogeneous wave equation.

Homogeneous wave equation:

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = 0$$

If $\psi(x,t) = X(x)\,T(t)$ (separation of variables)

Then we obtain

$$\underbrace{\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2}}_{+k^2} - \underbrace{\frac{1}{X}\nabla^2 X}_{k^2} = 0$$

Then for $X(x)$ we have

$$\nabla^2 X = -k^2 X$$

or $\nabla^2 X + k^2 X = 0$

$$\nabla^2 X + k^2 X = 0$$ Homogeneous Helmholz equation.

Now we want to solve the inhomogeneous Helmholz equation:

$$\nabla^2 \psi(\bar{x}) + k^2 \psi(\bar{x}) = -4\pi f(\bar{x}) \quad (1)$$

We are going to find the Green function to solve (1) considering that V is all the space.

The Green function $G(\bar{x}, \bar{x}')$ has to satisfy the Helmholz differential equation for the case in which $f(\bar{x}) = \delta(\bar{x} - \bar{x}')$.

$$\nabla^2 G(\bar{x}, \bar{x}') + k^2 G(\bar{x}, \bar{x}') = -4\pi\, \delta(\bar{x} - \bar{x}') \quad (2)$$

It can be shown that

$$\psi(\bar{x}) = \frac{1}{4\pi} \int_{\text{all space}} f(\bar{x}')\, G(\bar{x}, \bar{x}')\, d^3x'$$

for $G(\bar{x}, \bar{x}')$ solving $\nabla^2 G(\bar{x}, \bar{x}') + \lambda q(\bar{x}) G(\bar{x}, \bar{x}') = -4\pi\delta(\bar{x} - \bar{x}')$

We will find $G(\bar{x}, \bar{x}')$ solving ②:

- Since V is all space and space is isotropic G cannot depend on θ, φ, θ' or φ'.
- Then $G = G(R) \equiv G(r-r')$ $\quad r-r' = |\bar{r}-\bar{r}'|$.
- Then we will work in spherical coordinates but only the radial component will appear.

Then eq. (2) becomes:

$$\frac{1}{R}\frac{d^2(RG)}{dR^2} + k^2 G(R) = -4\pi\,\delta(R) \qquad (3)$$

For the homogeneous eq.

$$G(R) = \frac{A\,e^{ikR} + B\,e^{-ikR}}{R}$$

If $k \to 0$ (3) becomes Poisson's eq.

and $G(R) = \frac{1}{R}$ then $A+B=1$

Let's propose:

$$G_k^{\pm}(R) = \frac{e^{\pm ikR}}{R}$$

$+$ for $A=1$, $B=0$

$-$ for $A=0$, $B=1$

Then

$$\psi(r) = \frac{1}{4\pi}\int_{\text{all space}} dr' \, \frac{e^{\pm ik|r-r'|}}{|r-r'|} f(r')$$

Notice that $\psi(r) = \psi_k(r)$.

Now we want to solve:

$$\nabla^2 \psi(\bar{x},t) - \frac{1}{c^2}\frac{\partial^2 \psi(\bar{x},t)}{\partial t^2} = -4\pi f(\bar{x},t) \quad (5)$$

(Inhomogeneous wave equation).

We will find the Green function $G(\bar{x},\bar{x}',t,t')$ that solves:

$$\nabla^2 G(\bar{x},\bar{x}',t,t') - \frac{1}{c^2}\frac{\partial^2 G(\bar{x},\bar{x}',t,t')}{\partial t^2} = -4\pi\,\delta(\bar{x}-\bar{x}')\,\delta(t-t') \quad (6)$$

To solve the t part we will work in Fourier transformed space - Then consider:

$$\psi(\bar{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\bar{x},\omega)\, e^{-i\omega t}\, d\omega \quad (7)$$

$$j(\bar{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} j(\bar{x},\omega)\, e^{-i\omega t}\, d\omega \quad (8)$$

Replace (7) and (8) in (5):

$$\nabla^2 \psi(\bar{x},\omega) + \overbrace{\frac{\omega^2}{c^2}}^{k^2} \psi(\bar{x},\omega) = -4\pi f(\bar{x},\omega) \quad (9)$$

We have this for each ω.

Then (9) is Helmholz equation and I know then that:

$$\bar{\psi}(x,\omega) = \frac{1}{4\pi} \int d^3x' \underbrace{G(\bar{x},\bar{x}';\omega)}_{G_{k=\pm\frac{\omega}{c}}(\bar{x},\bar{x}')} f(\bar{x}',\omega) \quad (10)$$

with

$$G(\bar{x},\bar{x}',\omega) = G_k(\bar{x},\bar{x}') = \frac{e^{\pm ik|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|} \quad (11) \qquad k = \pm\omega/c$$

Now we need to obtain $G(\bar{x},\bar{x}',t,t')$.

We need to FT eq. 6 for the variable t (to its conjugate ω):

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \nabla^2 G(\bar{x},\bar{x}',\omega,t')\, e^{-i\omega t}\, d\omega \; -$$

applies only here

$$-\frac{1}{c^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2}{\partial t^2}\left[G(\bar{x},\bar{x}',\omega,t')\, e^{-i\omega t}\right] d\omega =$$

$$= -4\pi\,\delta(\bar{x}-\bar{x}')\,\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} e^{-i\omega t}\, e^{-i\omega t'}\, d\omega}_{\delta(t-t')} \qquad (12)$$

Let's rewrite (12):

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\nabla^2 G(\bar{x},\bar{x}',\omega,t') + \frac{\omega^2}{c^2} G(\bar{x},\bar{x}',\omega,t')\right] e^{-i\omega t} d\omega = -4\pi\,\delta(\bar{x}-\bar{x}')\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t'} e^{-i\omega t} d\omega \qquad (13)$$

Now propose that

$$G(\bar{x},\bar{x}',\omega,t') = \underbrace{G(\bar{x},\bar{x}',\omega)}_{h(\bar{x}')} g(t') \qquad (14)$$

Green function for Helmholz eq.

Plug (14) in (13):

For each value of ω we obtain:

$$\nabla^2 G(\bar{x},\bar{x}';\omega)\, g(t') + \frac{\omega^2}{c^2} G(\bar{x},\bar{x}';\omega)\, g(t') =$$

$$= -4\pi\, \delta(\bar{x}-\bar{x}')\, \frac{1}{\sqrt{2\pi}} e^{i\omega t'}$$

Then

equal (Helmholz eq.)

$$\left[\nabla^2 G(\bar{x},\bar{x}';\omega) + \frac{\omega^2}{c^2} G(\bar{x},\bar{x}';\omega)\right] g(t') =$$

$$= -4\pi\, \delta(\bar{x}-\bar{x}')\, \frac{1}{\sqrt{2\pi}} e^{i\omega t'}$$

equal

then:

$$g(t') = \frac{1}{\sqrt{2\pi}} e^{-i\omega t'} \qquad (15)$$

Then

$$G(\bar{x}, \bar{x}', \omega, t') = G(\bar{x}, \bar{x}', \omega) \frac{e^{-i\omega t'}}{\sqrt{2\pi}} =$$

$$= \frac{e^{\pm ik|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|} \frac{e^{-i\omega t'}}{\sqrt{2\pi}} \qquad (16)$$

We need to get rid of ω and obtain t.

To obtain $G(\bar{x}, \bar{x}', t, t')$ we need to AFT $G(\bar{x}, \bar{x}', \omega, t')$:

$$G^{\pm}(\bar{x}, \bar{x}', t, t') = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\pm i \frac{\omega}{c} |\bar{x} - \bar{x}'|}}{|\bar{x} - \bar{x}'|} e^{i\omega t'} e^{-i\omega t} d\omega$$

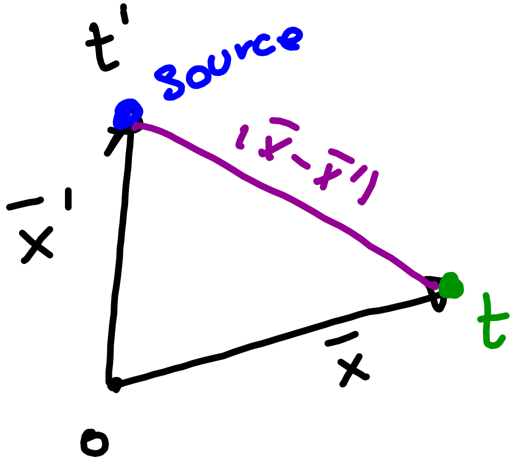
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega \left[t' - t \pm \frac{|\bar{x} - \bar{x}'|}{c} \right]}}{|\bar{x} - \bar{x}'|} d\omega \overset{*}{=} \frac{2\pi}{2\pi} \frac{\delta\left[t' - t \pm \frac{|\bar{x} - \bar{x}'|}{c} \right]}{|\bar{x} - \bar{x}'|}$$

$$* \quad \int_{-\infty}^{\infty} e^{-i\omega (t - t')} d\omega = 2\pi \, \delta(t - t')$$

Then we found that

$$G^{\pm}(\bar{x},\bar{x}',t,t') = \frac{\delta\left[t' - \left[t \mp \frac{|\bar{x}-\bar{x}'|}{c}\right]\right]}{|\bar{x}-\bar{x}'|}$$

G^{+} is called the retarded Green function.



It takes a time $t-t'$ for the effect of the point like source at $\bar{x}'$ and t' to reach $\bar{x}$.

If $t'<t$ then

$$t' = t - \frac{|\bar{x}-\bar{x}'|}{c} \quad \text{or}$$
$$t = t' + \frac{|\bar{x}-\bar{x}'|}{c}$$

On Tuesday 12/2 you will get in class the final exam to take home.

Bring a copy of the acknowledgment page of the online class evaluation to add to your class participation grade.