11/4

Reminder:

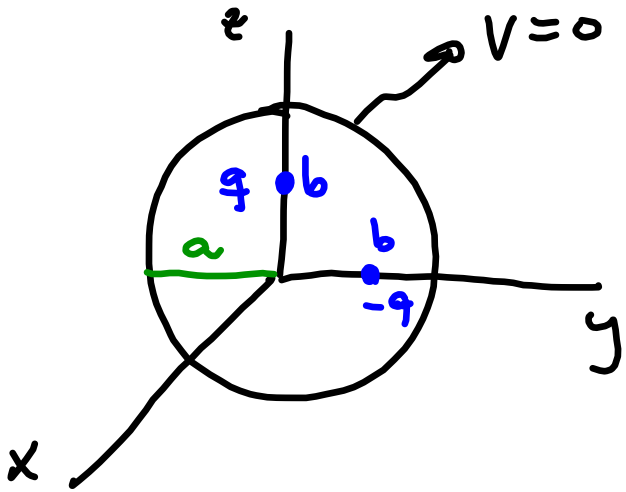
Second Midterm is on Thursday Nov. 13. Includes up to Hw #10 Due on Tuesday Nov. 11.

- One part is due in class on Nov. 13.
- Second part due on Tuesday Nov. 18.
  - Tensors: use of tensor notation to derive vector calculus expressions.
  - Covariant, contravariant components.
  - Use of metric tensor
  - Relativity.

- Differential equations:
  - Frobenius method.
  - Separation of variables in
    - cartesian
    - cylindric
    - spherical coordinates.

- You can use notes in class but not internet.
- At home can use whatever you want but you cannot talk with other people.

Example from last class (spherical coordinates without azimuthal symmetry):



Find $\phi(r,\theta,\varphi)$ for

$0 \leq r \leq a$.

- Since $\phi = \phi(r,\theta,\varphi)$ we need to use $Y_\ell^m(\theta,\varphi)$ in the solution.
- We will use the superposition principle adding by hand the potential due to q and $-q$.

We propose:

$$\phi(r,\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\, r^{\ell}\, Y_{\ell m}(\theta,\varphi)+$$

$$+\frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r_<^{\ell}}{r_>^{\ell+1}}\frac{1}{2\ell+1}Y_{\ell m}(\theta,\varphi)\,Y_{\ell m}^{*}(0,\varphi')$$

$$-\frac{q}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r_<^{\ell}}{r_>^{\ell+1}}\frac{1}{2\ell+1}Y_{\ell m}(\theta,\varphi)\,Y_{\ell m}^{*}\left(\frac{\pi}{2},\frac{\pi}{2}\right)$$

with $r_<$ ($r_>$) smaller (larger) between r and b.

To obtain $A_{\ell m}$ we use that $\phi(a,\theta,\varphi)=0$.

At $r=a$:

$$0=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{A_{\ell m}\,a^{\ell}+\frac{q}{4\pi\varepsilon_0}\frac{b^{\ell}}{a^{\ell+1}}\frac{1}{2\ell+1}\cdot\right.$$

$$\left.\cdot\left[Y^{*}_{\ell 0}(0,\varphi')-Y^{*}_{\ell m}\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right)\right]\right\}Y_{\ell m}(\theta,\varphi)$$

$$A_{\ell m}\,a^{\ell}+\frac{q}{4\pi\varepsilon_0}\frac{b^{\ell}}{a^{\ell+1}}\frac{1}{2\ell+1}\left(Y^{*}_{\ell 0}(0,\varphi')-Y^{*}_{\ell m}\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right)\right)=0$$

$$A_{\ell m}=\frac{q}{4\pi\varepsilon_0}\frac{b^{\ell}}{a^{2\ell+1}}\frac{1}{2\ell+1}\left(Y^{*}_{\ell m}\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right)-Y^{*}_{\ell 0}(0,\varphi')\right)$$

Notice that

$$Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{-im\varphi}$$

then

$$Y^*_{\ell m}(\theta,\varphi)=\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\underbrace{\theta}_{\frac{\pi}{2}})\,e^{-im\underbrace{\varphi}_{\frac{\pi}{2}}}$$

$\cos\theta=1 \quad P_\ell(1)=1$

$$Y_\ell^{0*}(\theta,\varphi')=\sqrt{\frac{(2\ell+1)}{4\pi}}\,P_\ell(\cos\theta)\overset{\phi}{=}\sqrt{\frac{(2\ell+1)}{4\pi}}$$

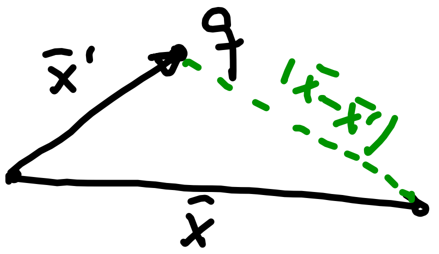
$$A_{\ell m}=\frac{q}{4\pi\varepsilon_0}\frac{b^\ell}{a^{2\ell+1}}\frac{1}{2\ell+1}\left[Y_\ell^{*m}\left(\frac{\pi}{2},\frac{\pi}{2}\right)-\sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m,0}\right]$$

Inhomogeneous differential equations:

Green functions (Ch. 10 in book)
See also Physics Today Dec. 2003 page 41

Example: Poisson's equation.

$$\nabla^2 \phi = -\frac{\rho(\bar{x})}{\varepsilon_0} \qquad (1)$$



if $q = 4\pi\varepsilon_0$

$$\phi(\bar{x}) = \frac{q}{4\pi\varepsilon_0 |\bar{x}-\bar{x}'|} = \frac{1}{|\bar{x}-\bar{x}'|} \qquad (2)$$

Also $\rho(\bar{x}) = q\,\delta(\bar{x}-\bar{x}') = 4\pi\varepsilon_0\,\delta(\bar{x}-\bar{x}')$ (3)

Plugging ② and ③ in ① we find that:

$$\nabla_x^2 \left(\frac{1}{|\bar{x}-\bar{x}'|} \right) = -\frac{4\pi \varepsilon_0 \, \delta(\bar{x}-\bar{x}')}{\varepsilon_0} = -4\pi \, \delta(\bar{x}-\bar{x}')$$

In infinite space we know that:

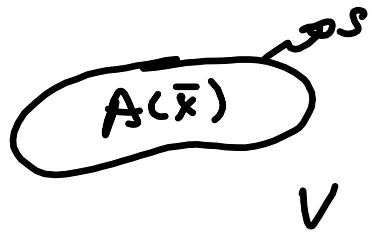
$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{all space}} \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|} \, d^3x'$$

elementary physics result.

What happens if $\rho(\bar{x}')$ is defined in a finite volume?

Detour: Green's theorem.

Consider a vector field $\bar{A}(\bar{x})$. We know that



$$① \int_V \bar{\nabla} \cdot \bar{A} \, d^3x' = \oint_S \bar{A} \cdot \hat{n}' \, da'$$

divergence theorem.

Green proposed that

$$\bar{A}(\bar{x}) = \phi(\bar{x}) \, \bar{\nabla} \psi(\bar{x}) \quad ②$$

ϕ and ψ are scalar functions of $\bar{x}$.

We want to plug (2) in (1). Notice that:

$$\bar{\nabla}_{x'} \cdot \bar{A} = \bar{\nabla}_{x'} \left(\phi(\bar{x}') \, \bar{\nabla} \psi(\bar{x}') \right) =$$

$$= \bar{\nabla}_{x'} \phi \, \bar{\nabla} \psi + \phi \, \nabla^2 \psi$$

$$\bar{A} \cdot \hat{n}' = \phi(\bar{x}') \underbrace{\bar{\nabla} \psi(\bar{x}') \cdot \hat{n}'}_{\frac{\partial \psi}{\partial n'}} = \phi \frac{\partial \psi}{\partial n'}$$

Then (1) becomes:

$$\int_V \left(\phi \nabla^2 \psi + \bar{\nabla} \phi \, \bar{\nabla} \psi \right) d^3 x' = \oint_S \phi \frac{\partial \psi}{\partial n'} \, da' \qquad (4)$$

Now exchange $\phi \longleftrightarrow \psi$ defining

$$\bar{A}(\bar{x}) = \psi \bar{\nabla} \phi$$

Replacing $\bar{A}$ in 1 we obtain:

$$\int_V (\psi \nabla^2 \phi + \bar{\nabla}\psi \, \bar{\nabla}\phi) \, d^3x' = \oint_S \psi \frac{\partial \phi}{\partial n'} \, da' \quad (5)$$

Now substract 5 from 4:

$$\boxed{\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3x' = \oint_S \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right) da'}$$

Green's theorem.

Now let's go back to our electrostatic problem.

Assume that $\psi(\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|}$

and

$\phi(x)$ is the electrical potential inside our volume of interest. Put ψ and ϕ in Green's theorem:

$\nabla^2_{x'} \frac{1}{|\bar{x}-\bar{x}'|} = \frac{\rho(\bar{x}')}{\varepsilon_0}$

$$\int_V \left(\phi(\bar{x}')(-4\pi\,\delta(\bar{x}-\bar{x}')) - \frac{1}{|\bar{x}-\bar{x}'|}\frac{\rho(\bar{x}')}{\varepsilon_0} \right) d^3x' =$$

$$= \oint_S \left(\phi(\bar{x}') \frac{\partial}{\partial n'}\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) - \frac{1}{|\bar{x}-\bar{x}'|}\frac{\partial \phi}{\partial n'} \right) da'$$

The $\delta(\bar{x}-\bar{x}')$ allows us to obtain $\phi(\bar{x})$:

$$-4\pi\,\phi(\bar{x}) + \frac{1}{\varepsilon_0}\int_V \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}\,d^3x' =$$

$$\oint_S \left[\phi\frac{\partial}{\partial n'}\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) - \frac{1}{|\bar{x}-\bar{x}'|}\frac{\partial\phi}{\partial n'}\right]da'$$

if $\bar{x}\in V$

0 otherwise.

Then for $\bar{x}$ inside V:

$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\bar{x}')}{|\bar{x}-\bar{x}'|}\,d^3x' + \frac{1}{4\pi}\oint_S\left[\frac{1}{|\bar{x}-\bar{x}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\bar{x}-\bar{x}'|}\right)\right]da'$$

Identical to the elementary result for $V\to\infty$ since $\oint_S \to 0$.

Now if we want to satisfy boundary conditions at the surface of V, we will have to add terms to $\psi = \frac{1}{|\bar{x}-\bar{x}'|}$

so that these terms can adjust the potential at the surface but have a zero laplacian inside the volume.

In other words the extra terms are going to be the potential of external charges that help to satisfy the b.c.'s at the boundary.

Then we propose:

$$\psi \to G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} + F(\bar{x},\bar{x}')$$

with

$$\nabla^2_{x'} F(\bar{x},\bar{x}') = 0 \quad \text{inside } V.$$

Now we put G instead of ψ in Green's theorem:

Notice that in V:

$$\nabla^2_{x'} G(\bar{x},\bar{x}') = \underbrace{\nabla^2_{x'} \frac{1}{|\bar{x}-\bar{x}'|}}_{-4\pi\delta(\bar{x}-\bar{x}')} + \underbrace{\nabla^2_{x'} F(\bar{x},\bar{x}')}_{0 \text{ in } V} = -4\pi\delta(\bar{x}-\bar{x}')$$

Then we obtain:

$$\int_V \Big(\phi \underbrace{\nabla^2 G}_{-4\pi\delta(\bar{x}-\bar{x}')} - G \underbrace{\nabla^2 \phi}_{+\frac{\rho(\bar{x}')}{\varepsilon_0}} \Big) d^3x' = \oint_S \Big(\phi \frac{\partial G}{\partial n'} - G \frac{\partial \phi}{\partial n'} \Big) da'.$$

then

$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \iint \rho(\bar{x}') G(\bar{x},\bar{x}') d^3x' + \frac{1}{4\pi} \oint_S \Big(G \frac{\partial \phi}{\partial n'} - \phi \frac{\partial G}{\partial n'} \Big) da'$$

The green function $G(\bar{x},\bar{x}')$ depends only on the geometry of the problem.