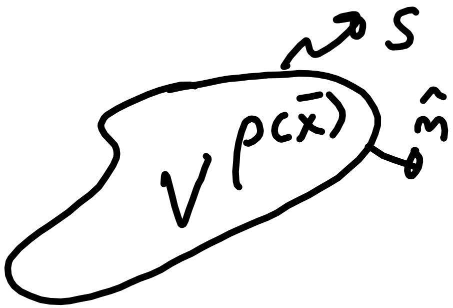
11/6:

Last time we found that

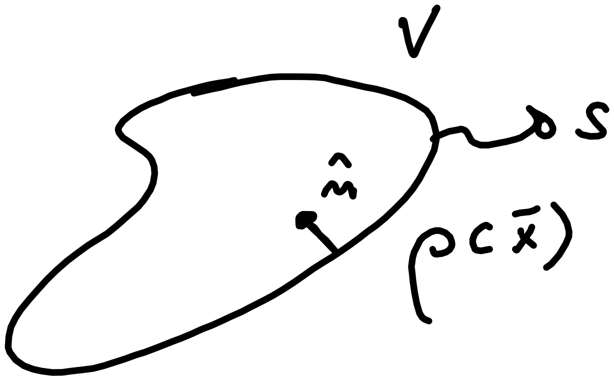
$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \rho(\bar{x}') \, G(\bar{x}, \bar{x}') \, d^3x' +$$

$$+ \frac{1}{4\pi} \oint_S \left(G \frac{\partial \phi}{\partial n'} - \phi \frac{\partial G}{\partial n'} \right) da'$$

$\hat{n}$: exterior normal



or

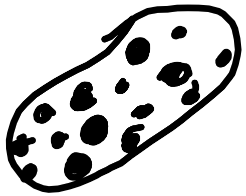


The $G(\bar{x},\bar{x}')$ will have to satisfy properties in agreement with the b.c:

1) Dirichlet b.c. (we know $\phi|_S$) then we chose $F(\bar{x},\bar{x}')$ so that $G(\bar{x},\bar{x}')\big|_S = 0$ so that the surface term $G\frac{\partial\phi}{\partial n} = 0$.

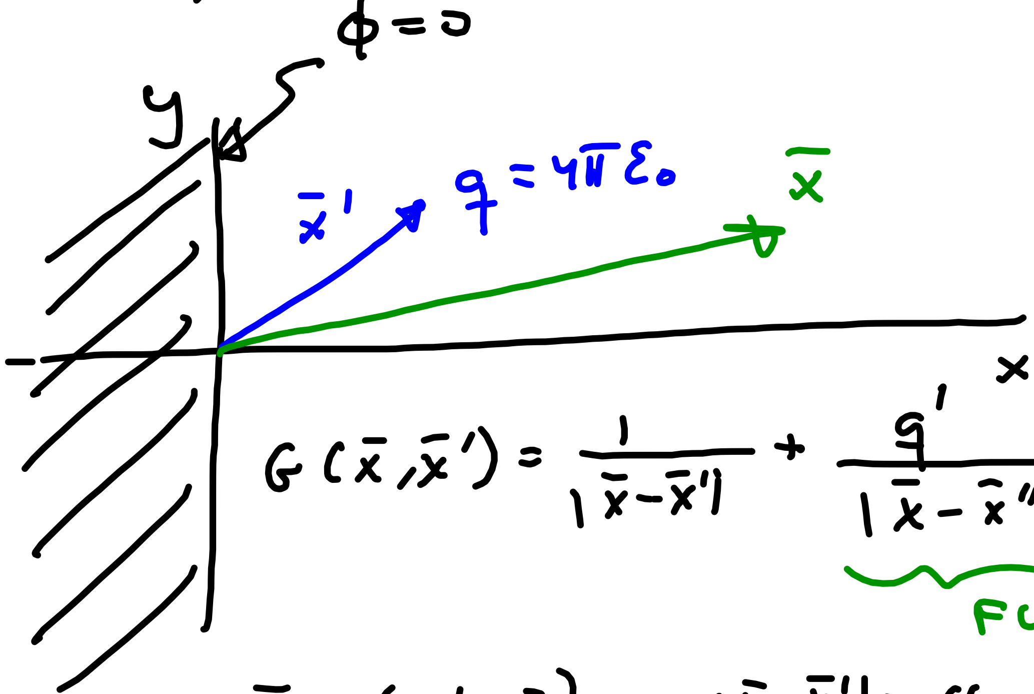
2) For von Neumann b.c. we know $E_n|_S = -\frac{\partial\phi}{\partial n}\big|_S$ then we need to request that $G(\bar{x},\bar{x}')$ is such that $\frac{\partial G}{\partial n}\Big|_S = -\frac{4\pi}{S}$ which vanishes when $S\to\infty$.

Then $G(\bar{x}, \bar{x}')$ is the potential of a point charge "unit" located at $\bar{x} = \bar{x}'$ that satisfies the b.c. Then $\phi(\bar{x})$ is obtained by integrating $\bar{x}'$ over all the locations inside V where $\rho(\bar{x})$ is finite.



$\rho(\bar{x})$ represented by point charges at x' of magnitude given by $\rho(\bar{x}')$

Examples: Find $G(\bar{x},\bar{x}')$ for Dirichlet b.c. for the right-hand-side of a plane containing z and y axis.



q' will have to be negative since all the other terms in G are positive

$$G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} + \underbrace{\frac{q'}{|\bar{x}-\bar{x}''|\,4\pi\varepsilon_0}}_{F(\bar{x},\bar{x}')}$$

For $\bar{x}' = (x', y', z')$ request $\bar{x}'' = (-x', y', z')$

$\phi = 0$ at $\bar{x} = (0, y, z)$

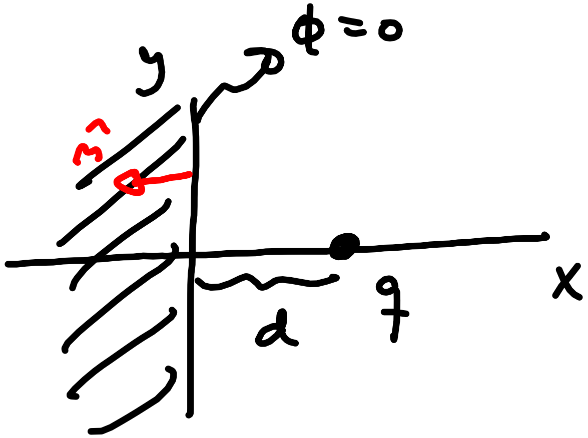
$$|\bar{x}-\bar{x}'| = ((x-x')^2 + (y-y')^2 + (z-z')^2)^{1/2}$$

Then

$$G(\bar{x}, \bar{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}$$

It vanishes at $\bar{x} = (0, y, z)$

Now let's use $G(\bar{x},\bar{x}')$ to solve a problem;



$$\rho(\bar{x}) = q\,\delta(x-d)\,\delta(y)\,\delta(z)$$

$$\hat{n} = -x$$

We can find $\phi(\bar{x})$ using $G(\bar{x},\bar{x}')$:

$$\phi(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{x}')\,G(\bar{x},\bar{x}')\,d^3x' - \underbrace{\frac{1}{4\pi}\oint_S \underbrace{\phi_s}_{0}\frac{\partial G}{\partial n'}\,da'}_{0}$$

$$= \int_0^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \frac{q\,\delta(x-d)}{4\pi\varepsilon_0}\left[\frac{1}{|\bar{x}-\bar{x}'|} - \frac{1}{|\bar{x}-\bar{x}''|}\right]$$

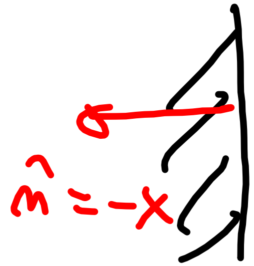
Using the δ's we obtain

$$\phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right]$$

Another addition to the problem:

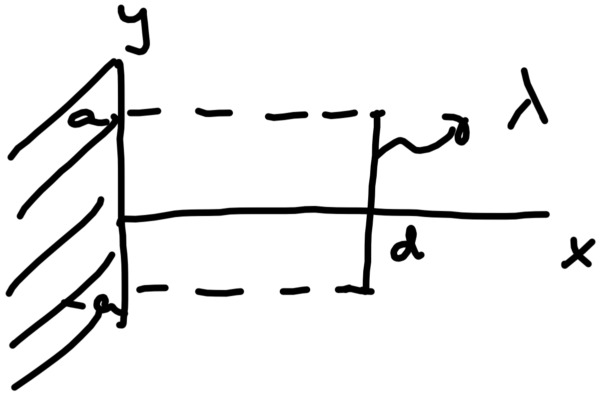
If $\phi_s = V$ then you will add

$$+\frac{V}{4\pi}\underbrace{\int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{\partial G}{\partial(-x')}\bigg|_{x=0}}_{4\pi}$$



So you will add V which you would see from the superposition principle.

Other variations



$$\rho(\bar{x}) = \lambda\,\delta(x'-d)\,\delta(z')\,\theta(y-a)\left[1-\theta(y+a)\right]$$

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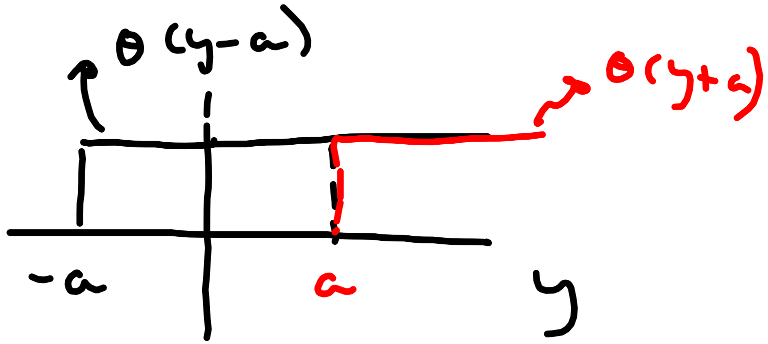
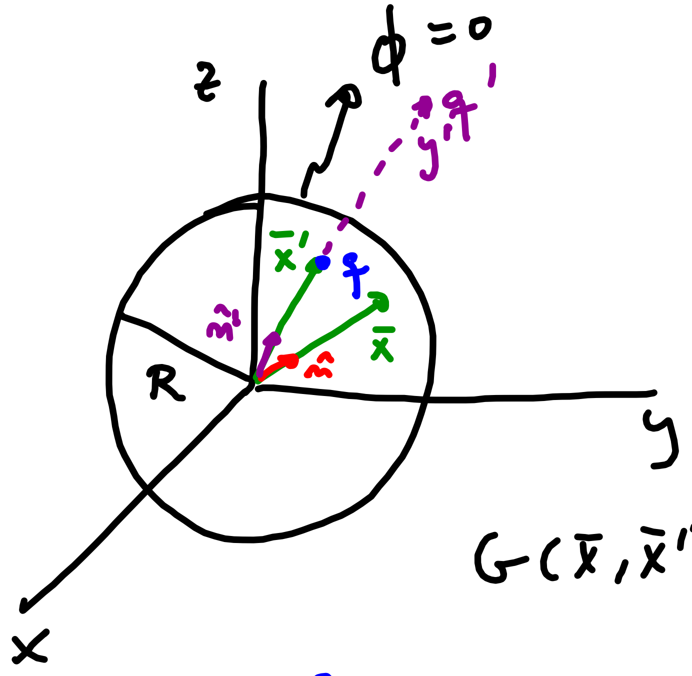


Green function for a spherical interphase.



$q = 4\pi\varepsilon_0$

V: $r \leq R$ (inside sphere)

$$G_D(\bar{x}, \bar{x}') = ?$$

$$G(\bar{x} - \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} + F(\bar{x}, \bar{x}')$$

$$G(\bar{x}, \bar{x}') = \frac{1}{|\bar{x} - \bar{x}'|} + \frac{q'}{4\pi\varepsilon_0 |\bar{x} - \bar{y}'|}$$

We need to find q' and $\bar{y}'$ and $|y'| > R$

At $|\bar{x}| = r$ we know that $G(\bar{r}, \bar{x}') = 0$.

$$0 = \frac{1}{|x\hat{n} - x'\hat{n}'|}\bigg|_{x=R} + \frac{q'}{4\pi\varepsilon_0 |x\hat{n} - y'\hat{n}'|}\bigg|_{x=R}$$

$$0 = \frac{1}{R\left|\hat{n} - \frac{x'}{R}\hat{n}'\right|} + \frac{q'}{4\pi\varepsilon_0 y' \left|\frac{R}{y'}\hat{n} - \hat{n}'\right|}$$

We will request that

$$\left|\hat{n} - \frac{x'}{R}\hat{n}'\right| = \left|\frac{R}{y'}\hat{n} - \hat{n}'\right| \quad (1) \quad \text{and} \quad \frac{1}{R} + \frac{q'}{4\pi\varepsilon_0 y'} = 0 \quad (2)$$

Let's consider ①

$$\underbrace{\hat{m}\cdot\hat{m}}_{1} - 2\,\hat{m}\cdot\hat{m}'\,\frac{x'}{R} + \frac{x'^2}{R^2}\,\underbrace{\hat{m}'\cdot\hat{m}'}_{1} =$$

$$= \frac{R^2}{y'^2}\,\underbrace{\hat{m}\cdot\hat{m}}_{1} - \frac{2R}{y'}\,\hat{m}\cdot\hat{m}' + \underbrace{\hat{m}'\cdot\hat{m}'}_{1}$$

$$\frac{x'}{R} = \frac{R}{y'} \Rightarrow \boxed{y' = \frac{R^2}{x'}} \quad ③$$

From ②:

$$\boxed{q'} = -\frac{4\pi\varepsilon_0 y'}{R} = \boxed{\frac{-4\pi\varepsilon_0 R}{x'}} \quad ④$$

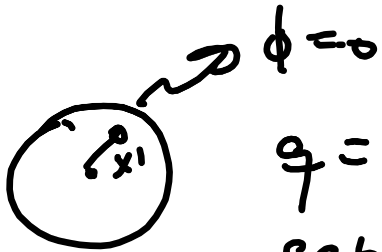
Replacing ③ and ④ in the expression for G we obtain:

$$\boxed{G(\bar{x},\bar{x}') = \frac{1}{|\bar{x}-\bar{x}'|} - \frac{R}{x'\left|\bar{x} - R^2\frac{\hat{n}'}{x'}\right|}}$$

It is useful to write $G(\bar{x},\bar{x}')$ in terms of $Y_{\ell m}(\theta,\varphi)$ using the fact that we know how to expand $1/|\bar{x}-\bar{x}'|$.

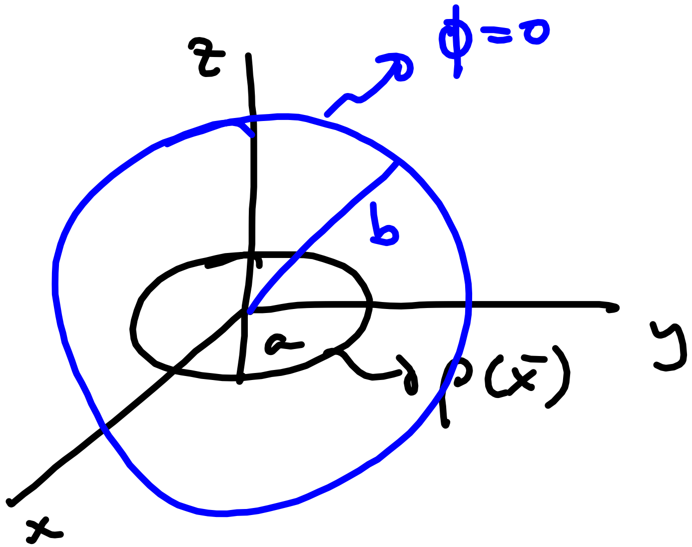
it only depends on the radius of the sphere and the position of the point charge at $\bar{x}'$.

In homework use the expansion for $\dfrac{1}{|\bar{x}-\bar{x}'|}$ in terms of $Y_{\ell m}(\theta,\varphi)$ that we found in class. Another way of getting this is by solving



$q = 4\pi\varepsilon_0$ at $\bar{x}'$ using separation of variables

Other problems that you can solve:



$$\rho(\bar{x}) = \frac{q}{2\pi a^2}\,\delta(r-a)\,\delta(\cos\theta)$$

you can use the Green function to calculate the potential due to $\rho(\bar{x})$.

Let's solve the problem using Green functions

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\bar{r}')\, G(\bar{r},\bar{r}')\, d^3r'$$

You will find that

image contribution
no singularity
inside V.

$$G(\bar{r},\bar{r}') = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^{\ell} r'^{\ell}}{b^{2\ell+1}}\right)$$

$$Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi')$$

$r_<$ ($r_>$) smaller (larger) between r and a.

Now

$$\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \frac{4\pi \quad q}{2\pi a^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} Y_{\ell m}(\theta,\varphi)$$

$$\int_0^b r'^2 dr' \, \delta(r'-a) \left(\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^{\ell} r'^{\ell}}{b^{2\ell+1}} \right) \int_{-1}^{1} d(\cos\theta') \delta(\cos\theta')$$

$$\underbrace{\int_0^{2\pi} d\varphi' \, \underbrace{Y^*_{\ell m}(\theta',\varphi')}_{\propto P_\ell^m(\cos\theta)\, e^{\mp im\varphi'}}}_{P_\ell(\cos\theta') \sqrt{\frac{2\ell+1}{4\pi}}\, 2\pi\, \delta_{m,0}}$$

$$= \frac{q}{a^2 \varepsilon_0} \sum_{\ell=0}^{\infty} \frac{P_\ell(\cos\theta)}{4\pi} \underbrace{\int_{-1}^{1} P_\ell(\cos\theta')\,\delta(\cos\theta')\,d\cos\theta'}_{P_\ell(0)}$$

$$\int_0^b r'^2\,dr'\,\delta(r'-a)\left(\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{r^\ell r'^\ell}{b^{2\ell+1}}\right) =$$

$$P_\ell(0) = \begin{cases} 0 & \text{for } \ell \text{ odd} \\ \text{for } \ell \text{ even} & P_{2j}(0) = \dfrac{(-1)^j (2j)!}{2^{2j}(j!)^2} \end{cases}$$

$$= \frac{q}{4\pi a^2 \varepsilon_0} \sum_{j=0}^{\infty} P_{2j}(\cos\theta) \frac{(-1)^j (2j)!}{2^{2j}(j!)^2} \; I$$

$$\mathbb{I} = \int_0^b r'^2 dr' \, \delta(r'-a) \left(\frac{r_<^{2j}}{r_>^{2j+1}} - \frac{r^{2j} r'^{2j}}{b^{4j+1}} \right)$$

Notice that this integral has 2 pieces:

For $r > a$ $\quad r_< = r'$ $\quad r_> = r$

$$\mathbb{I} = a^2 \left(\frac{a^{2j}}{r^{2j+1}} - \frac{r^{2j} a^{2j}}{b^{4j+1}} \right)$$

For $r < a$ $\quad r_< = r$ and $r_> = r'$

$$\mathbb{I} = a^2 \left(\frac{r^{2j}}{a^{2j+1}} - \frac{r^{2j} a^{2j}}{b^{4j+1}} \right)$$