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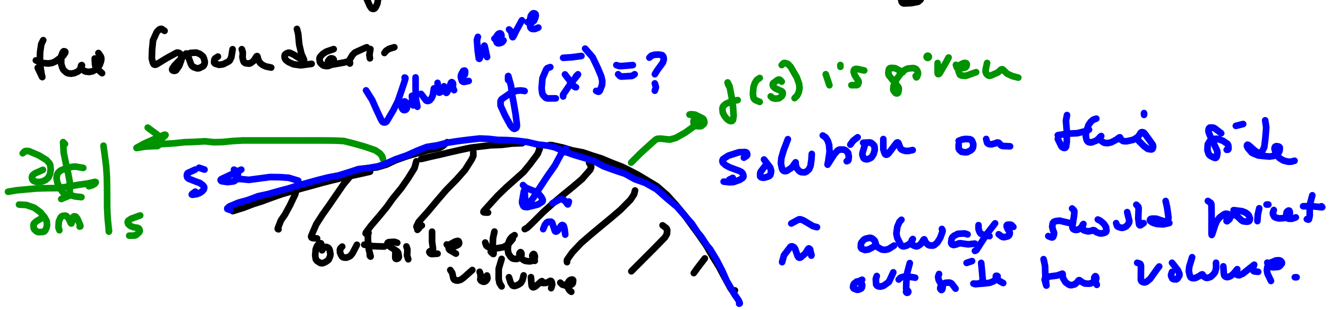
Differential Equations:

- Boundary conditions are needed to completely determine the solutions.
- In general we want to solve a problem inside a volume.

1) Cauchy B.C: $f(s)$ and $\left.\frac{\partial f}{\partial n}\right|_s$ are specified at the boundary



Cauchy b.c. are characteristic of problems defined by open surfaces (such as wave equation).

2) Dirichlet b.c: $f(s)$ is provided.

(close volumes for Poisson or Laplace's equation or in open surfaces for diffussion problems).

3) Newmann's b.c: $\left.\frac{\partial f}{\partial \hat{n}}\right|_S$ is provided.

For Poisson or Laplace in close surfaces.

In electrostatic

Dirichlet b.c. $\Rightarrow$ providing the potential on the surface boundaries

Neumann b.c. $\Rightarrow$ providing $\left.\frac{\partial \phi}{\partial \hat{n}}\right|_S = E_n\big|_S$ at the surface.

You have to provide one <u>or</u> the other – never both at the same time.

We will use mostly Dirichlet b.c. in our examples.

Separation of Variables.

A partial differential equation in n variables is separated into n ordinary differential equations. Each separation introduces a separation constant that is determined from the boundary conditions.

There are $n-1$ separation constants.

The form of the solutions to a PDE depends on the geometry of the surface where the b.c. are given.

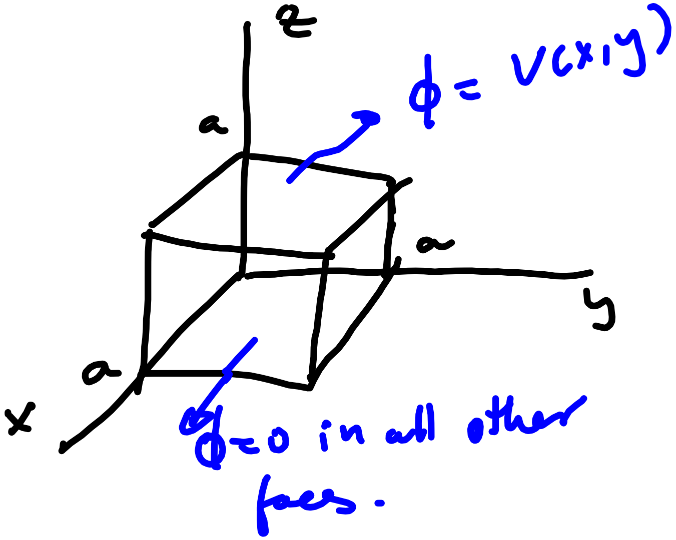
We will solve Laplace's equation:

$$\nabla^2 \phi = 0$$

in different geometries.

Cartesian coordinates:

$\nabla^2 \phi = 0$ with $\phi = \phi(x,y,z)$



We want to find $\phi(x,y,z)$ inside a cube of side a.
With $\phi(x,y,a) = V(x,y)$

and

$$\phi(0,y,z) = \phi(a,y,z) = \phi(x,0,z) = \phi(x,a,z) = \phi(x,y,0) = 0.$$

In this kind of geometry ϕ can be non-zero only on one face of the cube. To solve a problem with 2 or more faces set up at a non-zero potential you need to solve the problem for each face separately and use the superposition principle. (Example later).

Solution:

We propose that

$$\phi(x,y,z) = X(x)\,Y(y)\,Z(z) \quad (1)$$

this is very common in physical problems so one should try it and see whether it works.

Now plug ① in $\nabla^2\phi = 0$:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

Now divide by $\phi = XYZ$:

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\gamma^2 = \alpha^2 + \beta^2} = 0$$

α and β are the independent separation constants:

Then we have that

$$\frac{\partial^2 X}{dx^2} = -\alpha^2 X \Rightarrow X(x) \propto \cos\alpha x \text{ or } \sin\alpha x \text{ or } e^{\pm i\alpha x}$$

$$\frac{\partial^2 Y}{dy^2} = -\beta^2 Y \Rightarrow Y(y) \propto \cos\beta y \text{ or } \sin\beta y \text{ or } e^{\pm i\beta y}$$

$$\frac{\partial^2 Z}{\partial z^2} = \gamma^2 Z \Rightarrow Z(z) \propto \cosh\gamma z \text{ or } \sinh\gamma z \text{ or } e^{\pm\gamma z}$$

$\gamma^2 = \alpha^2 + \beta^2$

Then the most general solution of $\nabla^2\phi = 0$ is given by:

$$\phi(x,y,z) = \sum_{\alpha,\beta} (A_\alpha \cos\alpha x + B_\alpha \sin\alpha x)$$

$$(A_\beta \cos\beta y + B_\beta \sin\beta y)(A_{\alpha\beta} \cosh \gamma z +$$

$$+ B_{\alpha\beta} \sinh \gamma z) .$$

A_i and B_j, α, and β are determined from the b.c.

In our problem since $\phi(0,y,z) = \phi(x,0,z) =$

$= \phi(x,y,0) = 0$ then

$A_\alpha = A_\beta = A_{\alpha,\beta} = 0$ and

$$\phi(x,y,z) = \sum_{\alpha,\beta} \underbrace{B_\alpha B_\beta B_{\alpha\beta}}_{C_{\alpha\beta} = C_{n,m}} \sin\alpha x \sin\beta y \sinh\gamma z$$

$\gamma = \frac{\pi}{a}(n^2+m^2)^{1/2}$

Since $\phi(a,y,z) = 0 \Rightarrow \sin\alpha a = 0 \Rightarrow \alpha = \frac{n\pi}{a}$

$n = 1, 2, 3, \ldots$

Since $\phi(x,a,z) = 0 \Rightarrow \sin\beta a = 0 \Rightarrow \beta = \frac{m\pi}{a}$ $m = 1, 2, 3, \ldots$

$$\phi(x,y,z) = \sum_{n,m} C_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \sinh\gamma z$$

To find C_{nm} we use the last b.c. which is $\phi(x,y,a) = V(x,y)$.

$$\phi(x,y,a) = V(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin\frac{\pi n x}{a} \sin\frac{\pi m y}{a}$$

$$\times \sinh \frac{\pi}{a}(n^2+m^2)^{1/2} a$$

Since $\sin\frac{\pi n x}{a}$ and $\sin\frac{\pi m y}{a}$ are orthogonal functions in $[0,a]$ I can multiply both sides by $\sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{a}$ and integrate from 0 to a on x and y:

$$\int_0^a \int_0^a V(x,y) \sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{a}\, dx\, dy =$$

$$= \sum_{n,m=1}^{\infty} C_{n,m} \sinh \pi (n^2+m^2)^{1/2}$$

$$\underbrace{\int_0^a \sin\frac{n'\pi x}{a} \sin\frac{n\pi x}{a}\, dx}_{\frac{a}{2}\delta_{n,n'}} \underbrace{\int_0^a \sin\frac{m'\pi y}{a} \sin\frac{m\pi y}{a}\, dy}_{\frac{a}{2}\delta_{m,m'}}$$

$$= \frac{a^2}{4} C_{n'm'} \sinh \pi (n'^2+m'^2)^{1/2}$$

Then: (rename $n'=n$ and $m'=m$):

$$C_{nm} = \frac{\frac{4}{a^2}\int_0^a dx \int_0^a dy \; V(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}}{\sinh \pi (n^2+m^2)^{1/2}}$$

Now assume that $V(x,y) = V$:

$$C_{nm} = \frac{4V}{a^2 \sinh \pi (n^2+m^2)^{1/2} \, nm\pi^2} \underbrace{\overbrace{\int_0^a \sin\frac{n\pi x}{a}\, dx}^{-\frac{a}{n\pi}(-2)}}_{\substack{-2 \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even}}} \; \underbrace{\overbrace{\int_0^a \sin\frac{m\pi y}{a}\, dy}^{-\frac{a}{m\pi}(-2)}}_{\substack{-2 \text{ for } m \text{ odd} \\ 0 \text{ for } m \text{ even}}}$$

$$-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\Big|_0^a = -\frac{a}{n\pi}\left[(-1)^n - 1\right] = \frac{2a}{n\pi} \text{ if } n \text{ is odd } 0 \text{ if } n \text{ is even.}$$

Then

$$C_{nm} = \frac{16 V}{n m \pi^2 \sinh \pi (n^2 + m^2)^{1/2}}$$ if n and m are odd

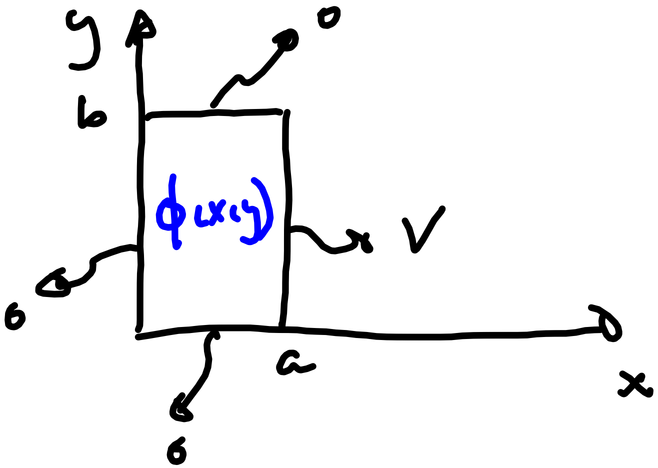
Then rename $n = 2j+1$ and $m = 2k+1$

and

$$\phi(x,y,z) = \frac{16 V}{\pi^2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sin\left[\frac{(2j+1)\pi x}{a}\right] \sin\left[\frac{(2k+1)\pi y}{a}\right]$$

$$\times \frac{\sinh \frac{\pi}{a}\left[(2j+1)^2 + (2k+1)^2\right]^{1/2} z}{\sinh \pi \left[(2j+1)^2 + (2k+1)^2\right]^{1/2}}$$

Example in 2D! (Cartesian):



$$\nabla^2 \phi = 0$$

$$\phi(x,y) = X(x)\, Y(y)$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{\alpha^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\alpha^2} = 0$$

I want harmonic solutions (periodic) along y.

Then

$$X(x) \propto e^{\pm \alpha x}$$

$$Y(y) \propto e^{\pm i \alpha y}$$

Then

$$\phi(x,y) = \sum_{\alpha} C_\alpha \sin \alpha y \sinh \alpha x$$

Since $\phi(x,b) = 0 \Rightarrow \sin \alpha b = 0 \Rightarrow \alpha = \frac{n\pi}{b}$

$n = 1, 2, \ldots$

$$\phi(x,y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}$$

To find C_n we use that $\phi(a,y) = V$

$$\underbrace{V \int_0^b \sin \frac{n'\pi y}{b}\, dy}_{\frac{2b}{n'\pi} \text{ for } n' \text{ odd},\ 0 \text{ for } n' \text{ even}} = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi a}{b} \overbrace{\int_0^b \sin \frac{n\pi y}{b} \sin \frac{n'\pi y}{b}\, dy}^{\frac{b}{2}\delta_{n,n'}}$$

$$C_n = \frac{4V}{n\pi \sinh \frac{\pi n a}{b}} \quad \text{for } n \text{ odd}$$

$$0 \text{ for } n \text{ even}$$

having $n = 2j+1$

$$\Phi(x,y) = \frac{4V}{\pi} \sum_{j=0}^{\infty} \frac{\sinh\left[\frac{(2j+1)\pi x}{b}\right] \sin \frac{(2j+1)\pi y}{b}}{(2j+1) \sinh \frac{\pi (2j+1) a}{b}}$$