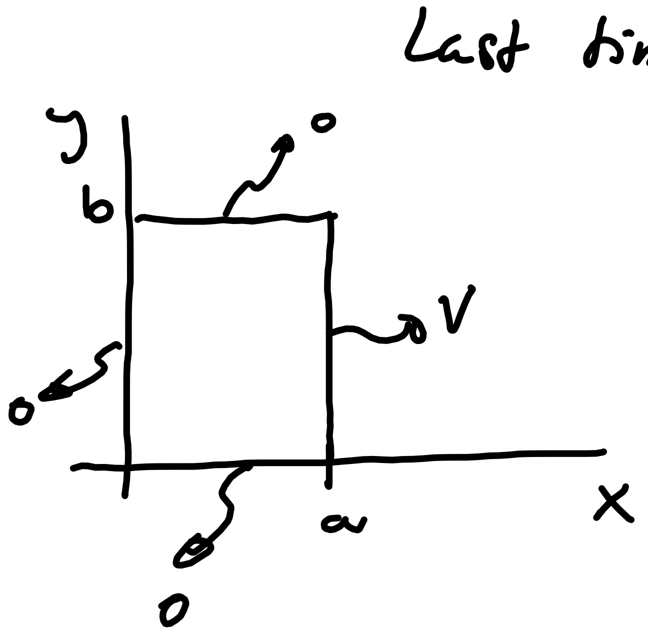
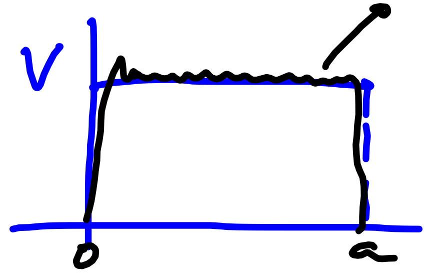
Last time: 10/21



V in Fourier representation:



as you add harmonics you reproduce V at every point approaching 0 or a

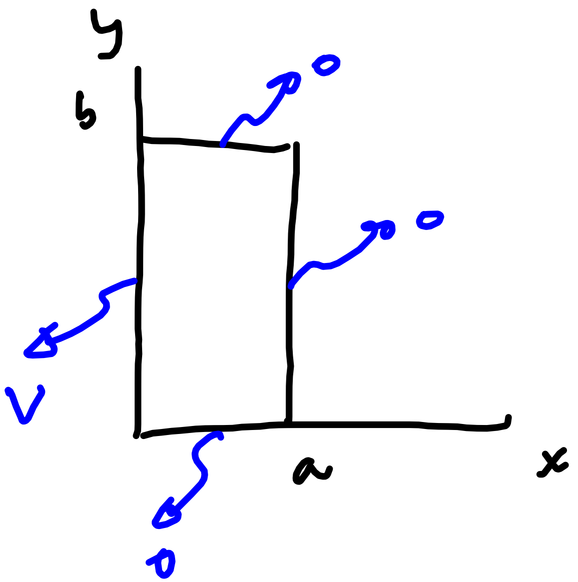
$$\phi(x,y) = \frac{4V}{\pi} \sum_{j=0}^{\infty} \frac{\sinh \frac{(2j+1)\pi x}{b} \sin \frac{(2j+1)\pi y}{b}}{(2j+1) \sinh \frac{\pi(2j+1)a}{b}}$$

0

At $(x,y) = (a,b)$ or $(a,0)$

$\phi = 0$.

Now let's solve this:



Way 1 (thinking about it):

$$\phi(x,y) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi y}{b} \cdot \sinh\frac{n\pi}{b}(a-x)$$

→ since $\phi(a,0)=0$

Now we use $\phi(0,y)=V$ to find A_n:

$$V = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi a}{b} \quad (1)$$

Multiply (1) by $\sin \frac{m\pi y}{b}$ and integrate both sides over y from 0 to b:

$$A_m = \frac{4V}{\sinh \frac{m\pi a}{b} \; m\pi} \quad \text{for } m \text{ odd}$$

0 for m even.

Then

$$\phi(x,y) = \frac{4V}{\pi} \sum_{j=0}^{\infty} \frac{\sinh \frac{(2j+1)\pi(a-x)}{b} \, \sin \frac{(2j+1)\pi y}{b}}{(2j+1) \sinh \frac{(2j+1)\pi a}{b}} \qquad (2)$$

Way 2) If you do not realize that you can write $\sinh\frac{(a-x)n\pi}{b}$ as part of the solution, you can propose:

$$\phi(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b}\left(A_n e^{\frac{n\pi x}{b}} + B_n e^{\frac{-n\pi x}{b}}\right)$$

$$\left[\text{or } \sum_{n=1}^{\infty} \sin\frac{n\pi y}{b}\left(C_n \sinh\frac{n\pi x}{b} + D_n \cosh\frac{n\pi x}{b}\right)\right.$$

We can find A_n and B_n (or C_n and D_n) using b.c:

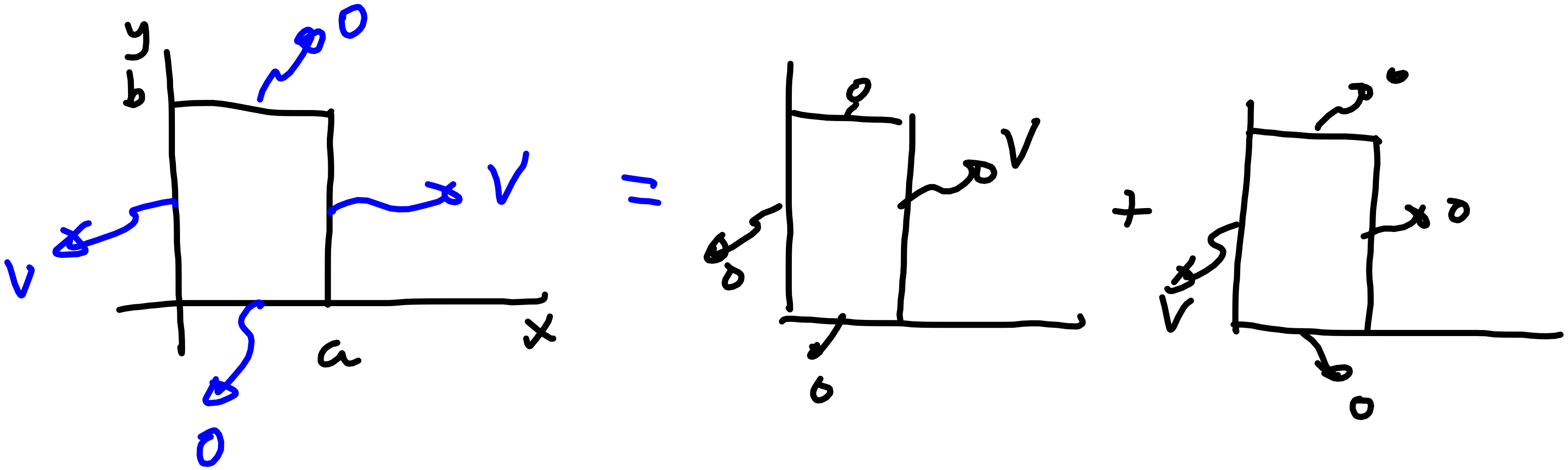
Asking that $\phi(a,y)=0$ and $\phi(0,y)=V$ you can obtain A_n and B_n and at the end the solution will be the same as (2)

If $\phi(a,y)=0 \Rightarrow A_n e^{\frac{n\pi a}{b}} + B_n e^{-\frac{n\pi a}{b}} = 0$

$$\therefore B_n = -A_n e^{\frac{2n\pi a}{b}} \quad (3)$$

Replace B_n by (3) and solve for A_n using that $\phi(0,y)=V$. Compare the result with (2):

Find $\phi(x,y)$:



solve each piece and add together since the principle of superposition is valid.

Then

$$\phi(x,y) = \frac{4V}{\pi} \sum_{j=0}^{\infty} \frac{\left[\sinh \frac{(2j+1)\pi x}{b} + \sinh \frac{(2j+1)\pi(a-x)}{b} \right]}{(2j+1) \sinh \frac{\pi(2j+1)a}{b}} \cdot \sin \frac{(2j+1)\pi y}{b}$$

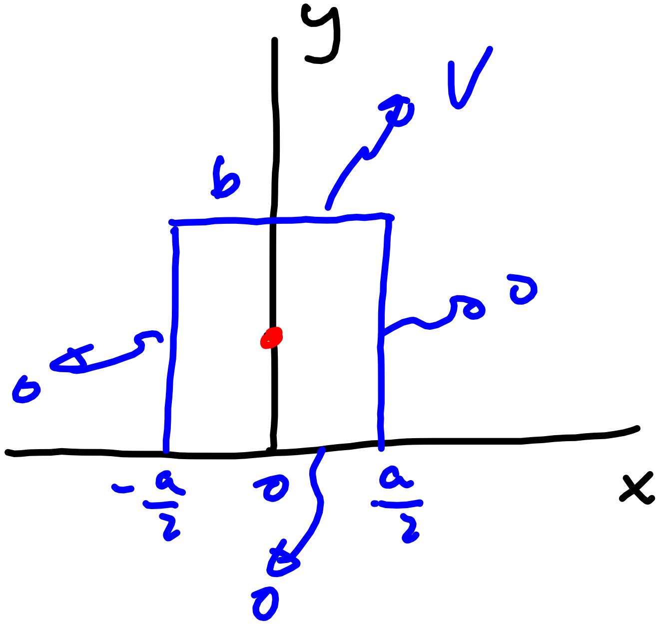
Notice that

$$\sinh[cx] + \sinh[c(a-x)] = 2 \sinh \frac{ca}{2} \cosh c\left(x - \frac{a}{2}\right)$$

Then

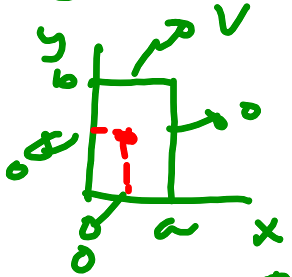
$$\phi(x,y) = \frac{8V}{\pi} \sum_{j=0}^{\infty} \frac{\sinh \frac{(2j+1)\pi a}{2b} \cosh \frac{(2j+1)\pi \left(x - \frac{a}{2}\right)}{b}}{(2j+1) \sinh \frac{\pi(2j+1)a}{b}} \sin \frac{(2j+1)\pi y}{b}$$

When do we need to use cosh x as solutions? For example if we use a different system of coordinates:



$\phi(0, \frac{b}{2})$: potential @ center.

From ⓪ we know that for



$\phi(\frac{a}{2}, \frac{b}{2})$ put @ center.

$$\phi(x,y) = \frac{4V}{\pi} \sum_{j=0}^{\infty} \frac{\sinh \frac{(2j+1)\pi y}{a}}{\sinh \frac{\pi(2j+1)b}{a}} \times \sin \frac{(2j+1)\pi y}{a} \qquad (4)$$

[We exchanged $x \leftrightarrow y$, $a \leftrightarrow b$ in ⓪].

With the new set of axis our solution should be:

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

Since $\cos\frac{n\pi a}{2a} = \cos\left(-\frac{n\pi a}{2a}\right) = \cos\frac{n\pi}{2} = 0.$

Then to find A_n we use that $\phi(x,b) = V$:

$$V = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}.$$

Multiply both sides by $\cos\frac{m\pi x}{a}$ and integrate over x from $-\frac{a}{2}$ to $\frac{a}{2}$.

You will get that:

$$A_m = \begin{cases} 0 & \text{if } m \text{ is even} \\ \dfrac{2V(-1)^j}{(2j+1)\pi \sinh\frac{(2j+1)b}{a}} & \text{if } m = 2j+1 \end{cases}$$

Notice that

$$\int_{-a/2}^{a/2} \cos\frac{m\pi x}{a}\,dx = \frac{a}{m\pi}\sin\frac{m\pi x}{a}\Big|_{-a/2}^{a/2} = \frac{a}{m\pi}\left[\sin\frac{m\pi}{2} - \sin\left(-\frac{m\pi}{2}\right)\right]$$

$$= \frac{2a}{m\pi}\sin\frac{m\pi}{2} \begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m = 1, 5, 9, \ldots \\ -1 & \text{if } m = 3, 7, 11, \ldots \end{cases}$$

Then

$$\phi(x,y) = \frac{4V}{\pi}\sum_{j=0}^{\infty} (-1)^j \frac{\cos\frac{(2j+1)\pi x}{a}\ \sinh\frac{(2j+1)\pi y}{a}}{(2j+1)\ \sinh\frac{(2j+1)\pi b}{a}} \qquad (5)$$

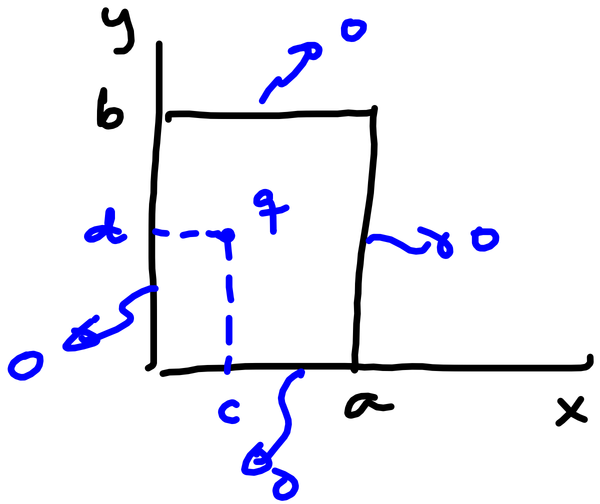
(5) is different from (4). However the potential at corresponding points in the box should be the same.

Then in (4) $\phi\left(\frac{a}{2},\frac{b}{2}\right) \overset{j=0}{\approx} \frac{4V}{\pi}\frac{\sinh\frac{\pi b}{2a}}{\sinh\frac{\pi b}{a}}$

In (5) $\phi\left(0,\frac{b}{2}\right) \overset{j=0}{\approx} \frac{4V}{\pi}\frac{\sinh\frac{\pi b}{2a}}{\sinh\pi b/a}$

Same as expected.

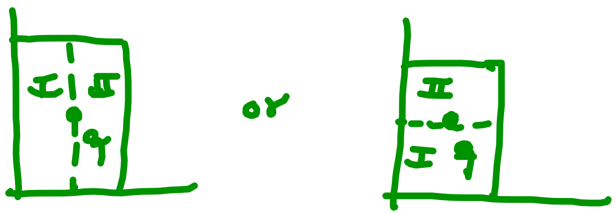
Charge inside the volume:



Find $\phi(x,y)$ inside the box.

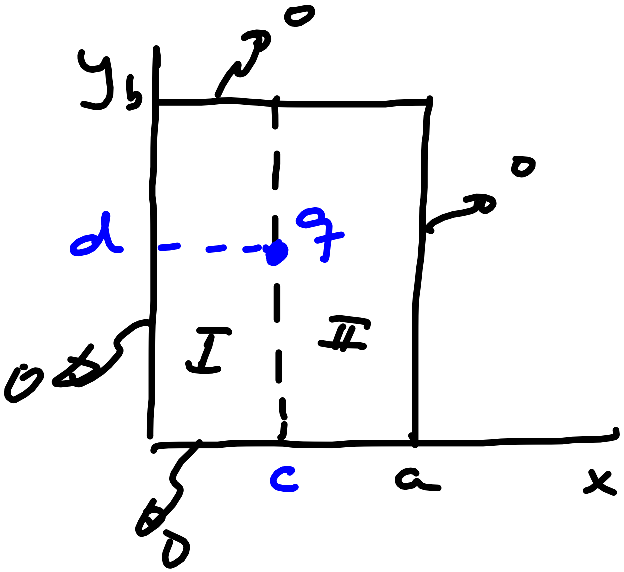
- We only know how to solve $\nabla^2\phi = 0$.
- So we partition our volume in two pieces where $\nabla^2\phi = 0$ and use boundary conditions where q is to find the coefficients.

We can chose:



One is obtained from the other by exchanging $x \leftrightarrow y$ $a \leftrightarrow b$ $c \leftrightarrow d$.

Let's consider partition 1:



$$\phi^{I}(x,y) = \sum_{m=1}^{\infty} A_m \sinh\frac{m\pi x}{b} \sin\frac{m\pi y}{b}$$

$$\phi^{II}(x,y) = \sum_{m=1}^{\infty} B_m \sinh\frac{m\pi(a-x)}{b} \sin\frac{m\pi y}{b}$$

I need to b.c. at $x = c$ to determine

A_m and B_m.

$$\phi^{I}\Big|_{x=c} = \phi^{II}\Big|_{x=c} \quad (1)$$

ϕ is always continuous across an interface since $\bar{\nabla}\times\bar{E} = 0$ $\int \bar{\nabla}\times\bar{E}\, ds = \int \bar{E}\cdot d\bar{l} = \phi_a - \phi_b$

We also know that

$\bar{\nabla}\cdot\bar{E} = \rho/\varepsilon_0$ then $E_n^{I} - E_n^{I} = \rho/\varepsilon_0$

$E_n = -\dfrac{\partial \phi}{\partial \hat{n}} = -\dfrac{\partial \phi}{\partial x}$ since $\hat{n} \parallel x$ in our problem.

ρ : density of charge in our "surface".

$\rho : q\,\delta(y-d)$ since we are already at $x=c$.

$$-\left.\frac{\partial \phi^{II}}{\partial x}\right|_{x=c} + \left.\frac{\partial \phi^{I}}{\partial x}\right|_{x=c} = \frac{q}{\varepsilon_0}\delta(y-d) \qquad \text{II}$$

From (I) we obtain:

$$A_m \sinh\frac{m\pi c}{b} = B_m \sinh\frac{m\pi(a-c)}{b}$$

$$\therefore \quad A_m = B_m \frac{\sinh\frac{m\pi(a-c)}{b}}{\sinh\frac{m\pi c}{b}}$$

From (II):

replace

$$\sum_{m=1}^{\infty} B_m \frac{m\pi}{b}\cosh\left(\frac{m\pi(a-c)}{b}\right)\sin\frac{m\pi y}{b} + \sum_{m=1}^{\infty} A_m \frac{m\pi}{b}\cosh\frac{m\pi c}{b}\sin\frac{m\pi y}{b} = \frac{q}{\varepsilon_0}\delta(y-d)$$

$$\sum_{n=1}^{\infty} B_n \frac{n\pi}{b} \sin\frac{n\pi y}{b}\left[\cosh\frac{n\pi(a-c)}{b} + \cosh\frac{n\pi c}{b}\,\frac{\sinh\frac{n\pi(a-c)}{b}}{\sinh\frac{n\pi c}{b}}\right] = \frac{q}{\varepsilon_0}\,\delta(y-\alpha)$$

Now multiply both sides by $\sin\frac{m\pi y}{b}$ and integrate from 0 to b. Then

$$\boxed{B_n = \frac{2q}{\pi\varepsilon_0}\sin\frac{n\pi\alpha}{b}\,\frac{\sinh\frac{n\pi c}{b}}{\sinh\frac{n\pi a}{b}}}$$

Then:

$$\phi^{I}(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{b}\,\frac{\sinh\frac{\pi n(a-c)}{b}}{\sinh\frac{\pi n a}{b}}\cdot\sin\frac{n\pi y}{b}\,\sinh\frac{\pi n x}{b} \quad \text{for } x \leq c.$$

$$\phi^{II}(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{b}\,\sin\frac{y n\pi}{b}\,\sinh\frac{\pi n(a-x)}{b}\,\frac{\sinh\frac{\pi n c}{b}}{\sinh\frac{\pi n a}{b}} \quad \text{for } x \geq c.$$

We can write ϕ as:

$$\phi(x,y) = \frac{2q}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi d}{c}\sin\frac{n\pi y}{b}.$$

$$\cdot\frac{\sinh\frac{\pi n x_<}{b}\,\sinh\frac{\pi n(a-x_>)}{b}}{\sinh\frac{\pi n a}{b}}$$

where $x_<$ $(x_>)$ is the smaller (larger) between c and x.