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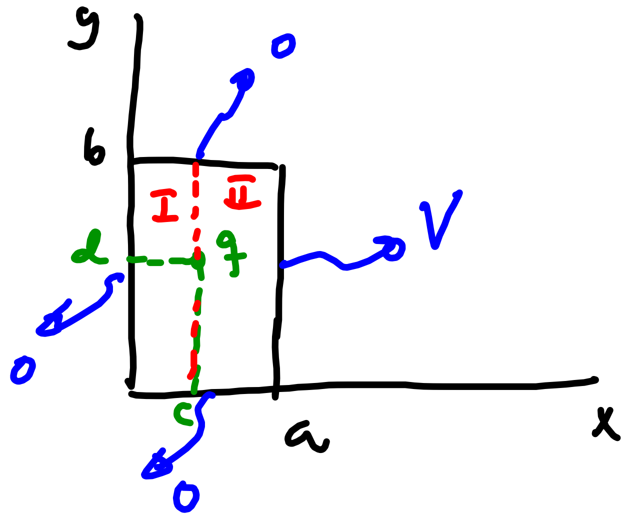
Principle of superposition.

Consider this problem:



Find $\phi(x,y)$ inside the box.

Hard way: • separate volume in 2 regions.

• Propose two solutions.

• use b.c.'s to find the coefficients.

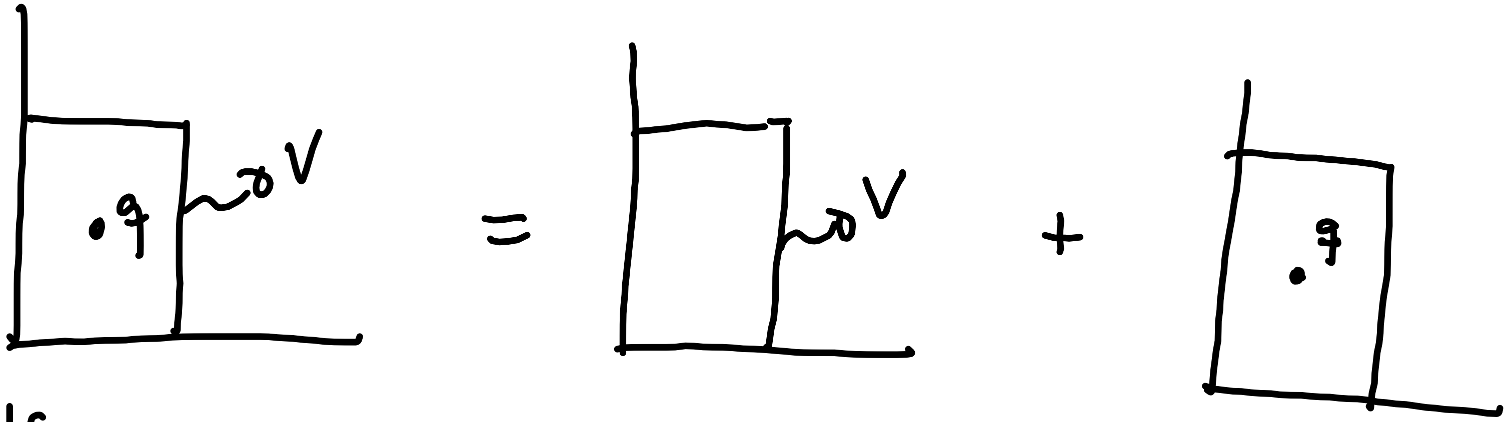
$$\phi^{I}(x,y) = \sum_{m=1}^{\infty} A_m \sinh\frac{m\pi x}{b} \sin\frac{n\pi y}{b}$$

$$\phi^{II}(x,y) = \sum_{m=1}^{\infty} \sin\frac{m\pi y}{b}\left[B_m e^{\frac{m\pi x}{b}} + C_m e^{-\frac{m\pi x}{b}}\right]$$

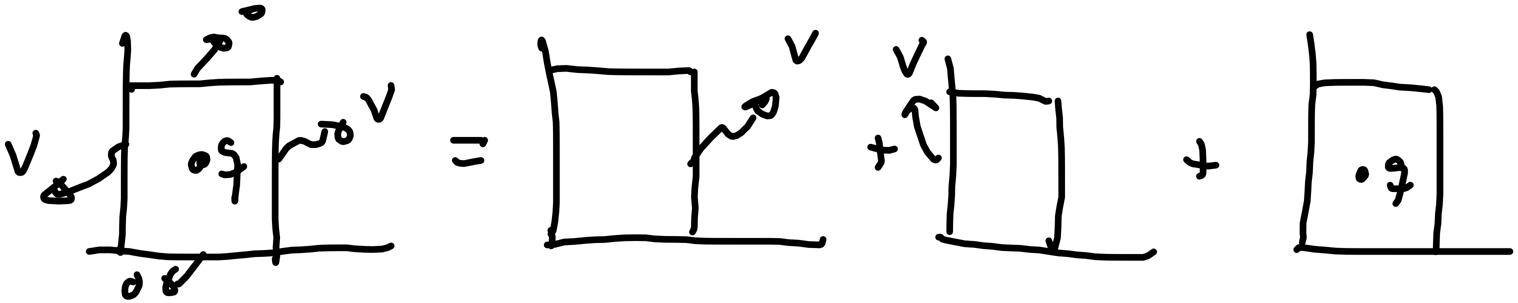
A_m, B_m and C_m can be determined by the 2 b.c.'s at $x=c$ and from $\phi(x=a,y)=V$.

But it is very tedious.

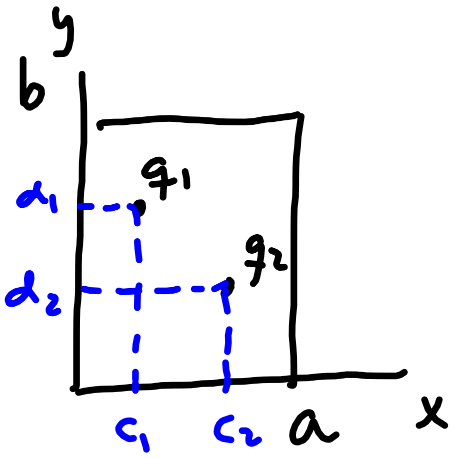
Then it is simpler if you use the principle of superposition:



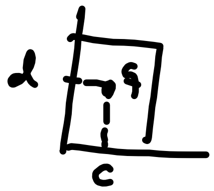
Also



Also:



The solution is the sum of the solution for



specializing for q_1 at (c_1, d_1) + q_2 at (c_2, d_2)

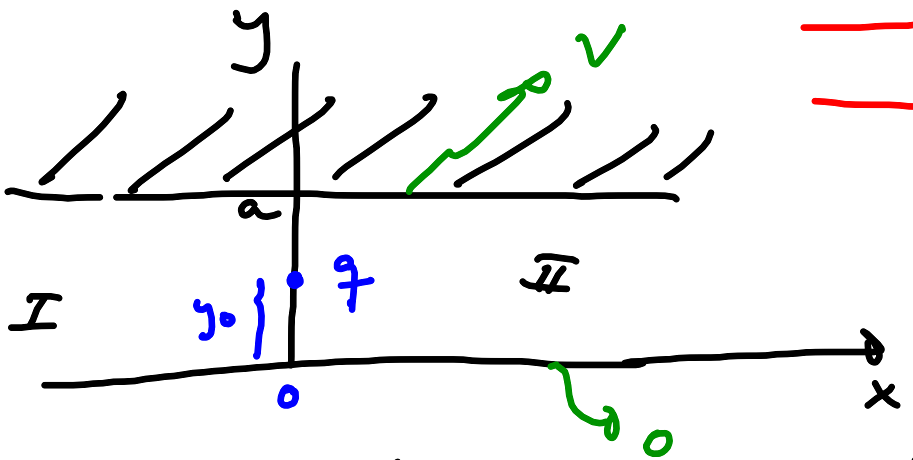
The sum of the 2 solutions will have 3 different forms:

ϕ^{I} for $0 \le x \le c_1$

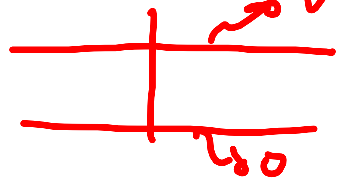
ϕ^{II} for $c_1 \le x \le c_2$

ϕ^{III} for $c_2 \le x \le a$.

Homework:



Caution:



$$\phi(x,y) \equiv \phi(y)$$

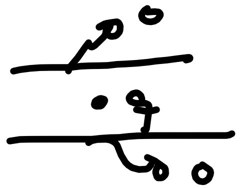
$$\nabla^2 \phi = 0 \Rightarrow$$

$$\frac{d^2Y}{dy^2} = 0$$

$$Y(y) = Ay$$

$Y(0) = 0$ then $A = \frac{V}{a}$

1) Solve



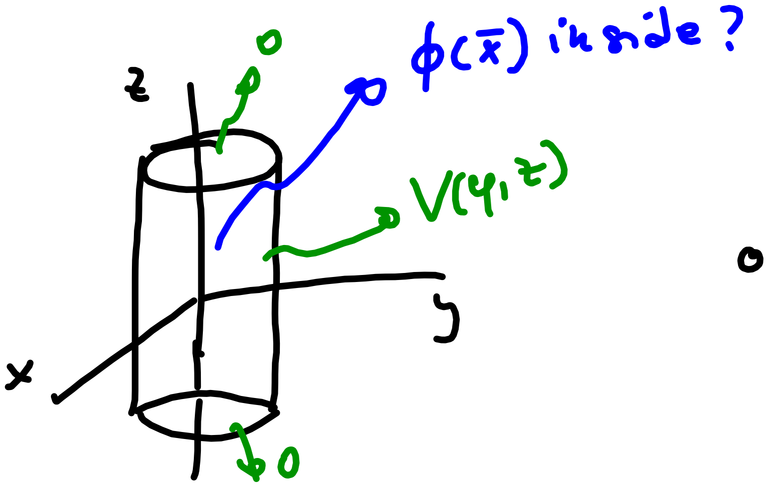
$$\phi^{I}(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{\pi n y}{a}\, e^{\frac{n\pi x}{a}}$$

$$\phi^{II}(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi y}{a}\, e^{-\frac{n\pi x}{a}}$$

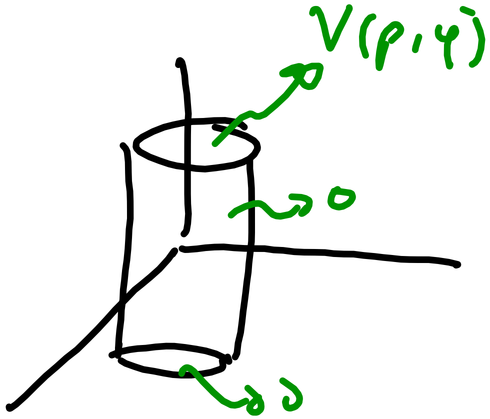
Find A_n and B_n from b.c.'s at $x=0$.

Cylindrical Coordinates.

If an electrostatic problem has b.c.'s given on a cylindrical surface then you should solve $\nabla^2 \phi = 0$ in cylindrical coordinates.



or



$$\nabla^2 \phi(\rho, \varphi, z) = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (1)$$

Separation of variables:

$$\phi(\rho, \varphi, z) = P(\rho)\, Q(\varphi)\, Z(z) \qquad (2)$$

Plug (2) in (1) and obtain 3 ODE

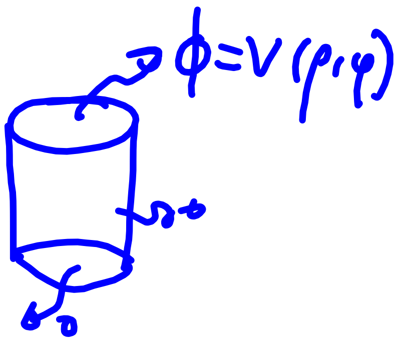
and 2 separation constants:

$$QZ\,\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{PZ}{\rho^2}\frac{d^2Q}{d\varphi^2}+PQ\,\frac{d^2Z}{dz^2}=0$$

Divide by PQZ:

$$\underbrace{\frac{1}{P}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho}\right)+\frac{1}{\rho^2Q}\frac{d^2Q}{d\varphi^2}}_{-k^2}+\underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{k^2}=0$$

If



we want non periodic solution for Z.

Then

$$\frac{d^2 Z}{dz^2} = k^2 Z \Rightarrow \boxed{Z(z) \propto e^{\pm kz}}$$

Then we are left with:

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) + \frac{1}{\rho^2 Q}\frac{d^2 Q}{d\varphi^2} = -k^2 \qquad (3)$$

Multiply (3) by ρ^2:

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho \frac{dP}{d\rho}\right) + \frac{1}{Q}\frac{d^2 Q}{d\varphi^2} = -k^2 \rho^2$$

Then

$$\underbrace{\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+k^2\rho^2}_{\nu^2} = \underbrace{-\frac{1}{Q}\frac{\partial^2 Q}{\partial\varphi^2}}_{+\nu^2}$$

Then

$$Q(\varphi) \propto e^{\pm i\nu\varphi} \qquad \text{periodic in } \varphi.$$

For $P(\rho)$ we have:

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+k^2\rho^2P-\nu^2P=0$$

$$\rho^2 \frac{d^2P}{d\rho^2} + \rho \frac{dP}{d\rho} + (k^2\rho^2 - \nu^2) P = 0$$

Divide by ρ^2:

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) P = 0$$

Define $x = k\rho \Rightarrow \rho = \frac{x}{k}$

$dx = k\,d\rho \Rightarrow d\rho = \frac{dx}{k}$

$$k^2 \frac{d^2P}{dx^2} + \frac{k^2}{x}\frac{dP}{dx} + \left(k^2 - \frac{\nu^2 k^2}{x^2}\right) P = 0$$

Divide by k^2:

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$$\boxed{\frac{d^2P}{dx^2} + \frac{1}{x}\frac{dP}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)P = 0}$$

Bessel's eq.

Solution is $P_\nu(x)$, Bessel function of order ν.

Once we find $P_\nu(x)$ we can solve $\nabla^2\phi = 0$:

$$\phi(\rho, \varphi, z) = \sum_{\nu, k} a_{\nu,k}\, P_{\nu,k}(\rho)\, \Phi_\nu(\varphi)\, Z_k(z)$$

Let's solve Bessel's equation:

Frobenius method (see 7.5 in book).

- Solves linear, second order, homogeneous ordinary diff. eqs. of the form:

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

where $y'' = \frac{d^2y}{dx^2}$ $\quad y' = \frac{dy}{dx}$

Most general solution is given by:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Propose a solution of the form:

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad (1)$$

and set equations to find a_λ and k.

If our equation is Bessel's eq:

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad (2)$$

From (1): $y'(x) = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}$ (3)

and $y''(x) = \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$ (4)

Plug 1, 3 and 4 in 2:

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)\, x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)\, x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda\, x^{k+\lambda+2} - \sum_{\lambda=0}^{\infty} a_\lambda\, m^2 x^{k+\lambda} = 0$$

Consider $\lambda = 0$. This gives us the lowest power of x in the solution which is x^k.

We need to ask that the coefficient of x^k vanishes.

$$a_0 k(k-1) + k a_0 - a_0 m^2 = 0$$

$$(k^2 - m^2) a_0 = 0 \implies \text{if } a_0 \neq 0 \text{ then}$$

$$k^2 = m^2 \text{ or } \boxed{k = \pm m}$$

indicial equation.

$k = m$ and $k = -m$ will give us the two independant solution $y_1(x)$ and $y_2(x)$.

Now consider the terms for $\lambda = 1$. The lowest power of x that appears is x^{k+1} whose coefficient has to vanish to solve the eq.

$$a_1 \left[(k+1)k + k + 1 - n^2 \right] = 0$$

$$a_1 \left[k^2 + 2k + 1 - n^2 \right] = 0$$

$$a_1 \left[(k+1)^2 - n^2 \right] = 0$$

$$a_1 (k+1-n)(k+1+n) = 0$$

Notice that if $k = n$ or $-n$ this does not vanish then we need that $a_1 = 0$.

If $n = -\frac{1}{2}$ this vanishes so we will have to do something different in that case.

You could use this as indicial eq. with $a_1 \neq 0$ but then you'll have to ask $a_0 = 0$. In BOTH cases you will obtain the same y_1 and y_2 solutions.

Now to find the other a_λ's let's consider $\lambda = j$ we will obtain the coefficients of x^{k+j} with $k=n$ we obtain:

$$a_j\left[(n+j)(n+j-1)+(n+j)-n^2\right]+a_{j-2}=0$$

Let's redefine $i=j-2$ $i+2=j$

$$a_{i+2}\left[(n+i+2)(n+i+1)+(n+i+2)-n^2\right]+a_i=0$$

Now call $i=j$:

$$a_{j+2} = - a_j \frac{1}{(j+2)(2n+j+2)}$$

Then you see that all the a_j's will have even j.

$$a_2 = -\frac{a_0}{2(2n+2)} = -\frac{a_0}{4(n+1)} = -\frac{a_0\, n!}{2^2\, 1!\, (n+1)!}$$

$$a_4 = -\frac{a_2}{4(2n+4)} = \frac{-a_2}{8(n+2)} = \frac{a_0\, n!}{2^4\, 2!\, (n+2)!}$$

in general:

$$a_{2p} = \frac{(-1)^p\, a_0\, n!}{2^{2p}\, p!\, (n+p)!}$$

So all the a_λ coefficients with even λ are obtained in terms of a_0 while all the coefficients with odd λ are 0.

Then

$$y_1(x) = a_0 x^n \left[1 - \frac{n!\, x^2}{2^2\, 1!\,(n+1)!} + \frac{n!\, x^4}{2^4\, 2!\,(n+2)!} + \cdots \right]$$

$$= a_0 \sum_{j=0}^{\infty} \frac{(-1)^j\, n!\, x^{n+2j}}{2^{2j}\, j!\,(n+j)!} = a_0 2^n n! \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(n+j)!} \left(\frac{x}{2}\right)^{n+2j}}_{J_n(x)}$$

$J_n(x)$: Bessel function of order n
(Ch. 14) for all $n \neq -\frac{1}{2}$).

If $k = -n$ you will obtain $y_2(x)$.