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Last time:

we used Frobenius method to solve Bessel's eq.

We found that $a_{j+2} = \dfrac{-a_j}{(j+2)(2n+j+2)}$ (*)

For $y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$

if $n<0$ and $j+2=-2n$ then (*) diverges

with $k = \pm n$.

If $n \neq -\frac{1}{2}$ we usually find $y_1(x)$ and $y_2(x)$ but if n is an integer then

When $n<0$ and $j+2=-2n$ then you obtain $J_n(x)$ in the usual way and something different has to be done to obtain $J_{-n}(x)$.

In Ch. 14 you can see that $J_{-n}(x) = (-1)^n J_n(x)$ for n integer.

Recipe: try Frobenius for solving second order ODE. If it works you are fine. Otherwise you need to try other method.

- Works for most physical cases.
- Check that $a_{\lambda+2}$ does not diverge for certain k's or λ's.

Laplace's equation in spherical coordinates.

$$\nabla^2 \phi = 0 \qquad \phi = \phi(r, \theta, \varphi)$$

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) +$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \qquad (1)$$

Propose:

$$\phi(r, \theta, \varphi) = \frac{U(r)}{r} P(\theta) Q(\varphi) \qquad (2)$$

Plug ② in ① and multiply by $\frac{r^2 \sin^2\theta}{UPQ}$:

$$\underbrace{r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{P r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]}_{m^2} +$$

$$+ \underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{-m^2} = 0$$

$\therefore \quad Q(\varphi) \propto e^{\pm i m \varphi} \qquad m = 0, 1, 2, 3, \ldots$

Notice that if the problem has azimuthal symmetry $\phi = \phi(r,\theta)$ then $Q(\varphi) = \mathbb{I} \Rightarrow m = 0$.

Now let's consider the (r, θ) part of the equation:

$$\frac{r^2 \sin^2\theta}{U}\frac{d^2U}{dr^2} + \frac{\sin\theta}{P}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) = m^2 \qquad (3)$$

Divide (3) by $\sin^2\theta$:

$$\underbrace{\frac{r^2}{U}\frac{d^2U}{dr^2}}_{\ell(\ell+1)} + \frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) = \frac{m^2}{\sin^2\theta}$$

and then

$$\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\ell(\ell+1) \qquad (4)$$

Then $U(r)$ can be found from:

$$\frac{r^2}{U}\frac{d^2U}{dr^2} = \ell(\ell+1)$$

$$\frac{d^2U}{dr^2} - \frac{\ell(\ell+1)}{r^2}U = 0 \Rightarrow U = A_\ell r^{\ell+1} + \frac{B_\ell}{r^\ell}$$

Then

$$\boxed{\frac{U}{r} \propto A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}}$$

Now consider ④ and multiply ④ by P:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

Let's define $x=\cos\theta$

$$dx=-\sin\theta\, d\theta$$

$$(1-x^2)^{1/2}=\sin\theta$$

$$\boxed{\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right)+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0} \qquad ⑤$$

Generalized Legendre equation.

Solutions are $P_\ell^m(x)$: generalized Legendre polynomials.

Let's solve (5) first for problems with azimuthal symmetry. So $m=0$ and (5) becomes:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\ell(\ell+1)P=0 \qquad (6)$$

Legendre equation. Has $P_\ell(x)$ as solution.

We will find $P_\ell(x)$ using Frobenius technique.

Notice that $x=\cos\theta$ and $0\le\theta\le\pi$ then $-1\le\cos\theta\le 1$ and then $\boxed{-1\le x\le 1}$.

So we need well behaved $P_\ell(x)$ for $-1\le x\le 1$.

We propose

$$P(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad (7)$$

Notice that $P(1)$ cannot diverge which means that $P(x)$ will have to be polynomials rather than infinite series.

Let's plug (7) in (6) and find k and a_λ:

$$\sum_{\lambda=0}^{\infty}\Big\{(k+\lambda)(k+\lambda-1)a_\lambda x^{k+\lambda-2} -$$

$$-\left[(k+\lambda)(k+\lambda+1)-\ell(\ell+1)\right]a_\lambda x^{k+\lambda}\Big\}=0$$

Now for $\lambda=0$ the lowest power of x is x^{k-2}

we need that its coefficient vanishes so:

$$k(k-1)a_0=0$$

if $a_0 \neq 0$ then $k=0$ or $k=1$.

For $\lambda = 1$ we find that $(k+1)k = 0$
if $a_1 \neq 0$, then $k = 0$ or -1.

- For a general λ, to make the coefficient of $x^{k+\lambda}$ vanish we need that:

$$(k+\lambda+2)(k+\lambda+2-1)a_{\lambda+2} = [(k+\lambda)(k+\lambda-1) - \ell(\ell+1)]a_\lambda$$

Then

$$a_{\lambda+2} = \frac{[(k+\lambda)(k+\lambda+1) - \ell(\ell+1)]}{(k+\lambda+2)(k+\lambda+1)} a_\lambda \qquad (8)$$

recurrence relation.

We can chose $a_0 \neq 0$ and $a_1 = 0$ or $a_0 = 0$ and $a_1 \neq 0$. The results will be the same.

If $a_0 \neq 0$ and $a_1 = 0$ then $k = 0$ or 1.

If $k = 0$:

$$P(x) = \sum_{j=0}^{\infty} a_{2j} x^{2j}$$

only even values of λ appear (even powers of x).

If $k = 1$:

$$P(x) = \sum_{j=0}^{\infty} a_{2j} x^{2j+1}$$

only even values of λ appear (odd powers of x).

Remember that $P(1)$ has to be finite.
Then we will see that only one of the two solutions will work depending on the value of ℓ.

Consider $\ell = 0$ and $k = 1$ in 8:

$$a_{\lambda+2} = \frac{(1+\lambda)\cancel{(\lambda+2)}}{(\lambda+3)\cancel{(\lambda+2)}} a_\lambda$$

then

$a_{\lambda+2} \neq 0$ for all λ's
and $P(1)$ diverges.

When $k=0$ we should get:

$$a_{\lambda+2} = \frac{[\lambda\cancel{(\lambda+1)}]\,a_\lambda}{\cancel{(\lambda+1)}(\lambda+2)} \qquad \neq 0 \text{ only for } a_0.$$

$$a_2 = a_4 = \dots = 0$$

Then

$$P_0(x) = a_0 x^{k+0} = a_0 x^0 = a_0 .$$

If you had selected $a_1 \neq 0$ you would have obtained that

$$P_0(x) = a_1$$ same result after normalization.

For $\ell \neq 0$ you will find that:

- $P_\ell(x)$ are polynomials of order ℓ.
- $P_\ell(x)$ is even if ℓ is even (all even powers of x)
- $P_\ell(x)$ is odd if ℓ is odd (all odd powers of x)

$P_\ell(x)$ are Legendre polynomials of order ℓ, with x^ℓ its higher power of x.

Normalization: $P_\ell(x=1)=1 \quad \forall \ell.$

Then $a_0 = 1$ (or $a_1 = 1$) for $\ell = 0$.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$\vdots$$

Properties of $P_\ell(x)$:

- Rodrigues' formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell$$

- Orthogonality: In $[-1,1]$:

$$\int_{-1}^{1} P_{\ell'}(x)\, P_\ell(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

- Then any well behaved function of x can be expanded in terms of $P_\ell(x)$ in $[-1,1]$.

Example:

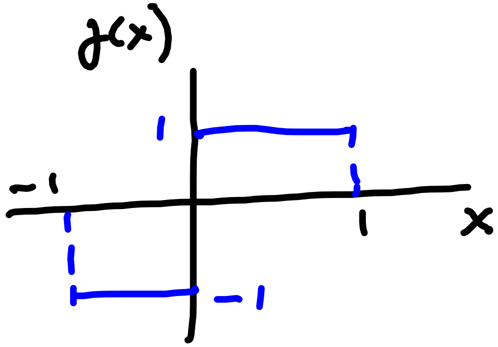
$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x) \qquad \text{for } -1 \le x \le 1$$

A_ℓ is obtained using orthogonality of $P_\ell(x)$;

$$\int_{-1}^{1} f(x)\, P_{\ell'}(x)\, dx = \sum_{\ell=0}^{\infty} A_\ell \underbrace{\int_{-1}^{1} P_{\ell'}(x)\, P_\ell(x)\, dx}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$\therefore \quad \boxed{A_{\ell'} = \frac{2\ell'+1}{2} \int_{-1}^{1} f(x)\, P_{\ell'}(x)\, dx} \qquad \underbrace{}_{A_{\ell'} \frac{2}{2\ell'+1}}$$

If $f(x) = \begin{cases} -1 & \text{for } -1 \le x < 0 \\ 1 & \text{for } 0 \le x \le 1 \end{cases}$



$$A_\ell = \frac{2\ell+1}{2}\left[\int_{-1}^{0}(-1)\,P_\ell(x)\,dx + \int_0^1 P_\ell(x)\,dx\right] = \begin{cases} 0 & \text{for } \ell \text{ even} \\ & \end{cases}$$

For ℓ odd: $P_\ell(x) = -P_\ell(-x)$

$$A_\ell = (2\ell+1)\int_0^1 P_\ell(x)\,dx = \frac{\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}}(2\ell+1)\,(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$m!! = m\,(m-2)\,(m-4)\cdots$

Going back to our original problem

$\nabla^2 \phi = 0$ with $\phi(r, \theta, \varphi) = \frac{U}{r} P(\theta) Q(\varphi)$

We have that $Q(\varphi) = 1$ because $m = 0$

and

$$\phi(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Where A_ℓ and B_ℓ will be determined by the b.c.