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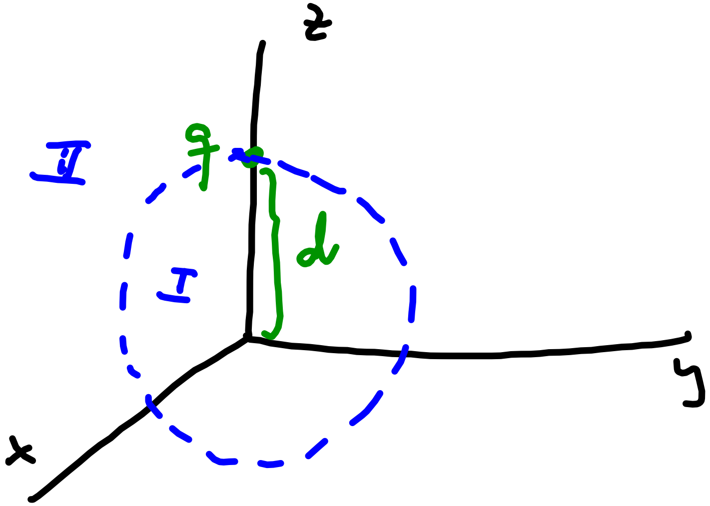
Example: Find the potential of a charge q in terms of $P_\ell(\cos\theta)$. This is the same as expanding $\frac{1}{|r-r'|}$ in terms of $P_\ell(\cos\theta)$.



$$\phi_q = \frac{q}{4\pi\varepsilon_0 |\bar{r}-\bar{d}|}$$

Express in terms of $P_\ell(\cos\theta)$

- We divide space in two regions where $\nabla^2\phi = 0$.
- Propose solutions.
- Adjust coefficients using b.c. at $r=d$.

$$\phi^{I}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \qquad 0 \le r \le d$$

$$\phi^{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad d \le r \le \infty$$

At $r = d$:

$\hat{n} = \hat{r}$ in this case

① $\phi^{I}(r,\theta) = \phi^{II}(r,\theta)$

② $-\left.\frac{\partial \phi^{II}}{\partial r}\right|_{r=d} + \left.\frac{\partial \phi^{I}}{\partial r}\right|_{r=d} = \frac{\sigma}{\varepsilon_0} = \frac{q\,\delta(\cos\theta - 1)}{2\pi d^2 \varepsilon_0}$

What is σ? $\sigma = \frac{q\,\delta(\cos\theta-1)}{2\pi d^2}$

$$q = \int_{surf} \sigma\, ds = \int_{-1}^{1} \cancel{d^2}\, d(\cos\theta) \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \frac{q\,\delta(\cos\theta-1)}{\cancel{2\pi}\, \cancel{d^2}} = q \overbrace{\int_{-1}^{1} \delta(\cos\theta-1)\, d(\cos\theta)}^{1}$$

From ① we find that

$$\sum_{\ell=0}^{\infty} A_\ell d^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{d^{\ell+1}} P_\ell(\cos\theta)$$

Due to orthogonality of $P_\ell(\cos\theta)$ the coefficients for each ℓ have to be equal:

$$A_\ell d^\ell = \frac{B_\ell}{d^{\ell+1}} \Rightarrow \boxed{A_\ell = \frac{B_\ell}{d^{2\ell+1}}} \quad (*)$$

Now
$$\phi^{I}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{d^{2\ell+1}} r^{\ell} P_\ell(\cos\theta)$$

$$\phi^{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Then replacing in (2):

$$-\left.\frac{\partial \phi^{II}}{\partial r}\right|_{r=d} + \left.\frac{\partial \phi^{I}}{\partial r}\right|_{r=d} = \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)B_\ell}{d^{\ell+2}} + \frac{\ell B_\ell}{d^{\ell+2}}\right] P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{(2\ell+1)B_\ell}{d^{\ell+2}} P_\ell(\cos\theta) = \frac{q\,\delta(\cos\theta - 1)}{2\pi d^2 \varepsilon_0} \quad (3)$$

Multiply both sides of (3) by $P_{\ell'}(\cos\theta)$ and integrate over $\cos\theta$ from -1 to 1:

$$\sum_{\ell=0}^{\infty} \frac{(2\ell+1)B_\ell}{d^{\ell+2}} \underbrace{\int_{-1}^{1} P_{\ell'}(\cos\theta)\, P_\ell(\cos\theta)\, d(\cos\theta)}_{\frac{2}{2\ell+1}\delta_{\ell,\ell'}} =$$

$$= \frac{q}{2\pi d^2 \varepsilon_0} \underbrace{\int_{-1}^{1} P_{\ell'}(\cos\theta)\, \delta(\cos\theta - 1)\, d(\cos\theta)}_{P_{\ell'}(1) \equiv 1}$$

Define:
$\ell' = \ell$

$$\frac{\cancel{(2\ell+1)}B_\ell}{d^{\ell+2}} \frac{2}{\cancel{2\ell+1}} = \frac{q}{2\pi d^2 \varepsilon_0} \Rightarrow \boxed{B_\ell = \frac{q\, d^\ell}{4\pi \varepsilon_0}} \quad (4)$$

Plugging (4) in (*) we obtain:

$$\boxed{A_\ell = \frac{q}{4\pi\varepsilon_0 d^{\ell+1}}}$$

Then:

$$\begin{cases} \phi^{I}(r,\theta) = \dfrac{q}{4\pi\varepsilon_0} \displaystyle\sum_{\ell=0}^{\infty} \frac{r^\ell}{d^{\ell+1}} P_\ell(\cos\theta) \\ \phi^{II}(r,\theta) = \dfrac{q}{4\pi\varepsilon_0} \displaystyle\sum_{\ell=0}^{\infty} \frac{d^\ell}{r^{\ell+1}} P_\ell(\cos\theta) \end{cases}$$

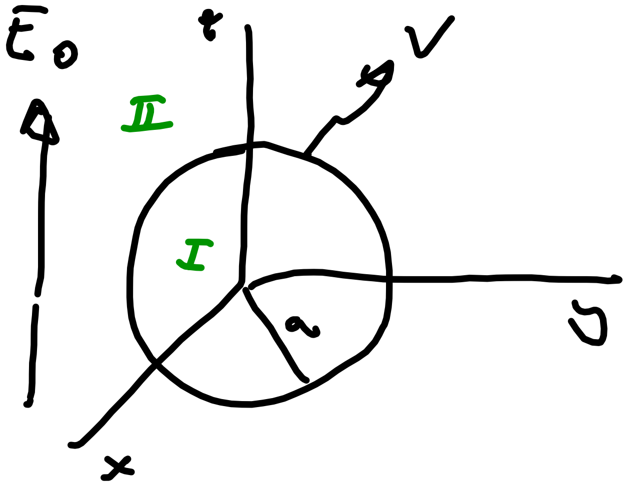
or

$$\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

where $r_<$ ($r_>$) is the smaller (larger) between r and d.

Another example with azimuthal symmetry;

Application: 15.2.11.a (problem in the book)



Find $\sigma(\theta)$ on the sphere.

$$\phi^{I}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

we know that

$\phi^{I}(a,\theta) = V$ and due to

continuity $\phi^{I}(r,\theta) = V$.

and:

$$\phi^{II}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) - E_0 r P_1(\cos\theta) \quad (*)$$

Notice that for ϕ^{II} we add the potential that generates σ_0.

$$\phi_{E_0} = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta).$$

At $r = a$: $\quad \phi^{I} = \phi^{II} = V \qquad \mathrm{I}$

$$-\frac{\partial\phi^{II}}{\partial r} + \frac{\partial\phi^{I}}{\partial r} = \frac{\sigma(\theta)}{\varepsilon_0} \qquad \mathrm{II}$$

$$\sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta) = V \equiv V \underbrace{P_0(\cos\theta)}_{1}$$

Then $A_0 = V$ and $A_\ell = 0$ for all $\ell > 0$.

From ① we get $\phi^{II} = V$.

$$\sum_{\ell=0}^{\infty} \frac{B_\ell}{a^{\ell+1}} P_\ell(\cos\theta) - E_0 a P_1(\cos\theta) = V P_0(\cos\theta)$$

For $\ell = 0$: $\quad \frac{B_0}{a} = V \implies \boxed{B_0 = Va}$

For $\ell = 1$:

$$\frac{B_1}{a^2} - E_0 a = 0 \implies \boxed{B_1 = E_0 a^3}$$

For $\ell > 1$:

$$\frac{B_\ell}{a^{\ell+1}} = 0 \implies \boxed{B_\ell = 0}$$

$$\phi^{I}(r,\theta) = V$$

$$\phi^{II}(r,\theta) = \frac{aV}{r} + \frac{E_0 a^3}{r^2} \underbrace{P_1(\cos\theta)}_{\cos\theta} - E_0 r \underbrace{P_1(\cos\theta)}_{\cos\theta}$$

Then

$$\boxed{\sigma(\theta)} = \varepsilon_0 \left[-\frac{\partial \phi^{II}}{\partial r}\bigg|_a + \underbrace{\frac{\partial \phi^{I}}{\partial r}\bigg|_a}_{0} \right] =$$

$$= \frac{aV}{a^2} + \frac{2E_0 a^3}{a^3}\cos\theta + E_0 \cos\theta =$$

$$\boxed{= \left(\frac{V}{a} + 3E_0\cos\theta\right)\varepsilon_0}$$

Problems without azimuthal symmetry.
(Now $m \neq 0$).

We need to use the solutions of the generalized Legendre equation:

$P_\ell^m(\cos\theta)$ Associated Legendre polynomials.

$$P_\ell^m(\cos\theta) = (-1)^m \, (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$

with $-\ell \le m \le \ell$.

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

$P_\ell^m(x)$ are orthogonal in ℓ in the interval $[-1,1]$:

SAME m!

$$\int_{-1}^{1} P_{\ell'}^m(x)\, P_\ell^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell,\ell'}$$

We need to remember that since $m \neq 0$ now $\Phi(\varphi) = e^{\pm i m \varphi}$.

Then the solution to the angular part of $\nabla^2 \phi = 0$ is

$$P(\theta)\, Q(\varphi) \propto P_\ell^m(\cos\theta)\, e^{\pm i m \varphi}$$

Let's define:

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{i m \varphi}$$

↳ spherical harmonics

ℓ and m are integers.

$$\left\{ \begin{array}{l} -\ell \le m \le \ell \\ 0 \le \ell \le \infty \end{array} \right.$$

Properties of $Y_{\ell m}(\theta,\varphi)$:

$$Y_{\ell,-m}(\theta,\varphi) = (-1)^m Y^*_{\ell m}(\theta,\varphi)$$

$\{Y_{\ell m}\}$ form an orthonormal basis in the interval $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$.

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell,\ell'}\,\delta_{m,m'}$$

This means that any well behaved function $f(\theta,\varphi)$ can be expanded in terms of $Y_{\ell m}(\theta,\varphi)$:

$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\varphi)$$

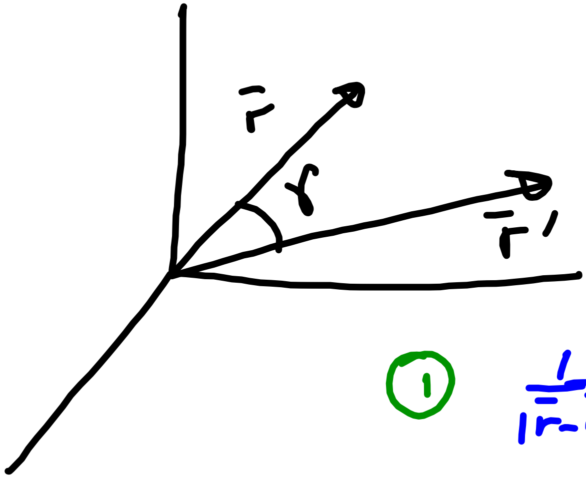
with:

$$A_{\ell m} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; Y_{\ell m}^{*}(\theta,\varphi) f(\theta,\varphi)$$

Then the general solution of $\nabla^2 \phi = 0$ is given by:

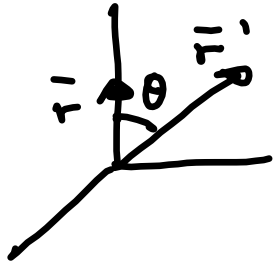
$$\boxed{\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\varphi)}$$

Expansion of $\frac{1}{|\vec{r}-\vec{r}\,'|}$ in $Y_{\ell m}(\theta,\varphi)$:



We found that

$$\frac{1}{|r-r'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)$$



Then in general

$$\frac{1}{|\bar{r}-\bar{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma) \qquad (1)$$

Theorem of addition of spherical Harmonics:

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi') \qquad (2)$$

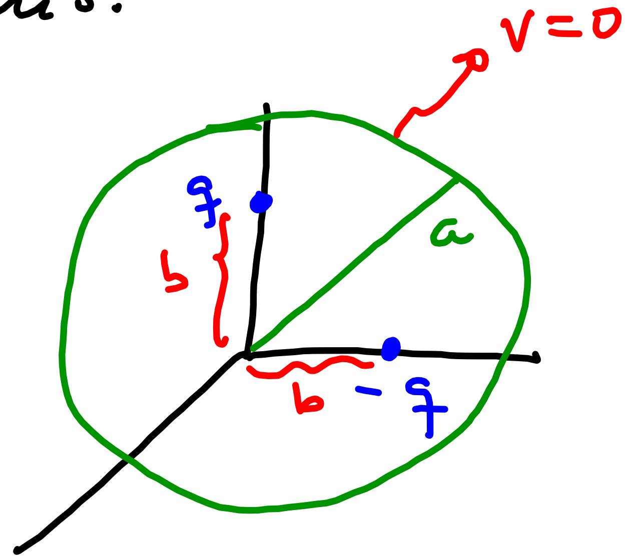
Plugging ② in ① we obtain:

$$\frac{1}{|\bar{r}-\bar{r}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}^{*}(\theta,\varphi) \cdot Y_{\ell m}(\theta',\varphi')$$

Then you know $\phi(r,\theta,\varphi)$ for a charge q at $\bar{r}'$ in terms of $Y_{\ell m}(\theta,\varphi)$ since

$$\phi(r,\theta,\varphi) = \frac{q}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}$$

Now you can solve a problem like this:



Find $\phi(r,\theta,\varphi)$ for $0 \leq r \leq a$.

We will use the principle of superposition so that

$r_<$ is smaller between r and b.

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} r^\ell Y_{\ell m}(\theta,\varphi) +$$

$$+ \frac{q}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{r_<^\ell}{r_>^{\ell+1}} \frac{1}{2\ell+1} Y_{\ell m}^*(\theta,\varphi)\left[Y_{\ell m}(0,\varphi') - Y_{\ell m}\left(\tfrac{\pi}{2},\tfrac{\pi}{2}\right)\right]$$