Tensor transformations

10/7

(Chapter 4).

We already know how tensors of any rank transform going from S to S':

$$V'^{i} = \frac{\partial x'^{i}}{\partial q^{j}} V^{j} \quad (1)$$

$\{x'^{i}\}$ in S'

$\{q^{j}\}$ in S

- What happens with the derivatives of tensors?

Consider ψ an scalar.

$\bar{\nabla}_i \psi = \partial_i \psi$ rank 1 tensor.

Then

$$\partial'_j \psi' = \frac{\partial q^i}{\partial x'^j} \partial_i \psi$$ covariant vector transformation.

And

$$\partial^j \psi = g^{ij} \partial_i \psi$$ contravariant form.

What happens if we want to take the derivative of a tensor of higher rank than 0?

- Consider a vector:

$$\frac{\partial V'^i}{\partial q^j} \overset{①}{=} \underbrace{\frac{\partial^2 x'^i}{\partial q^j \partial q^k}}_{} V^k + \frac{\partial x'^i}{\partial q^k} \frac{\partial V^k}{\partial q^j}$$

this does not look like the way a tensor should transform.

To fix this problem a covariant derivative can be defined:

$$V^k_{;j} = \frac{\partial V^k}{\partial q^j} + V^\mu \Gamma^k_{j\mu}$$

All this is a mixed tensor of rank 2.

$$\Gamma^k_{j\mu} = \bar{\varepsilon}^k \cdot \frac{\partial \bar{\varepsilon}_\mu}{\partial q^j}$$

Christoffel symbol of the second kind

You can read more about this in Ch. 4 if interested.

Tensors in Relativity.

Ch. 4 and also in ch. 17-8-9:

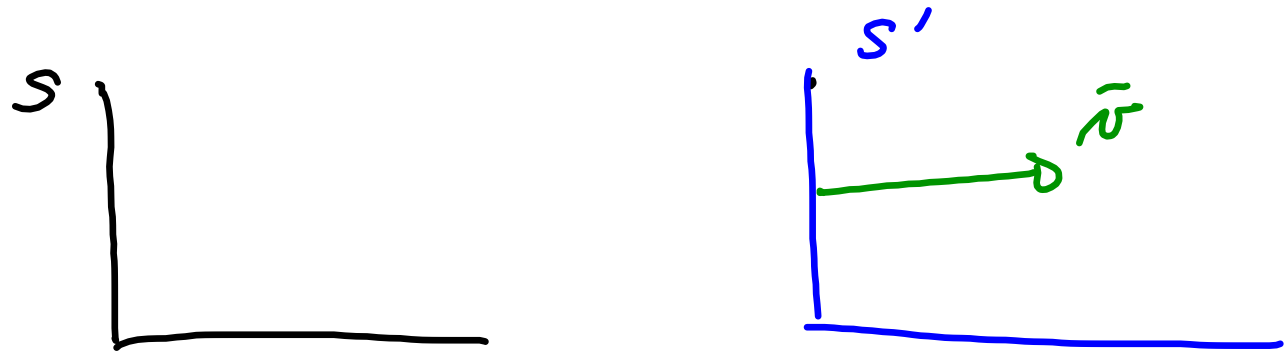
Book uses SI system (it has lots of c's appearing)
I will use gaussian system (it is "cleaner").

Relativity:

- The laws of Physics have to be invariant under Lorentz transformation.
  Maxwell's equations satisfy this but Newton's mechanics does not.

- Invariance under space translations and time translations due to isotropy of space-time.
- Invariance under 3-D rotations in real space (isotropic).
- Main postulate of relativity: the speed of light c is the same in all reference frames.

Lorentz transformations:



Since c is the same in S and S' if we turn on a spherical wave of light at $t=t'=0$ and $\bar{r}=\bar{r}'=0$ then under a Lorentz transformation

$$c^2t^2-x^2-y^2-z^2 = ct'^2-x'^2-y'^2-z'^2.$$

Then at time t $|\bar{r}| \neq |\bar{r}'|$ but the invariant is $c^2t^2 - |\bar{r}|^2$.

Let's define:

$$ds^2 = (ct)^2 - x^2 - y^2 - z^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

$$\parallel$$

$$dr_\mu \, dr^\mu = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \, dx^i \, dx^j = g_{ij} \, dx^i \, dx^j$$

Then

$$g_{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Also some people prefer $\bar{\varepsilon}_0 = (i, 0, 0, 0)$ then they define

$$g_{ij} = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

You need to choose one convention and use it.

- In our convention the spatial basis vectors are imaginary.
- In the $\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ convention the time basis vector is imaginary.
- Also instead of defining $x^0 = ct$ you can define $x^4 = ict$

then

$$ds^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 \text{ invariant}$$

and $g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ Now $\begin{array}{l} \varepsilon_1 = (1,0,0,0) \\ \vdots \\ \varepsilon_4 = (0,0,0,1) \end{array}$ all real.

In our convention $g_{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ and then

$$g^{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \quad \text{since} \quad g_{ij}\, g^{jk} = \delta_i{}^k$$

Now the tensors of rank 1 are called 4-vectors and are given by:

$$\underbrace{x^\mu = (x^0, \vec{x})}_{}$$ contravariant 4-vector

x^0: scalar in 3D

$\vec{x}$: vector in 3D.

$$x_\mu = g_{\mu\nu} x^\nu = (x_0, -\bar{x})$$

covariant 4-vector

Notice that $\mu = 0, 1, 2, 3$

Greek letters are used when the indeces go from 0 to 3.

When using the 1 to 4 convention latin letters are used for indices.

Scalar products:

$$x_\mu x^\mu = g_{\mu\nu} x^\nu x^\mu = (x^0)^2 - |\bar{x}|^2$$

Derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\partial_0, \vec{\nabla}\right)$$

covariant derivative

$$\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$$

then

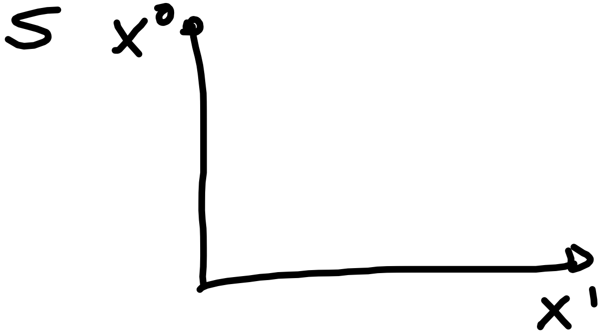
$$\partial^\mu = g^{\mu\nu}\partial_\nu = \left(\partial_0, -\vec{\nabla}\right)$$

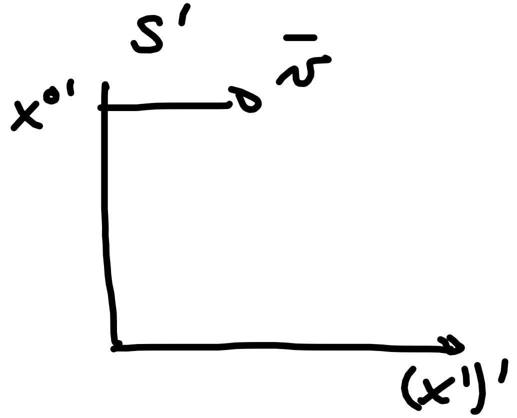
contravariant derivative.

$$\partial^2 = \Box = \partial_\mu \partial^\mu = \partial_{01}^2 - \nabla^2$$

d'Alambertian

Lorentz' transformation:





Using that for a spherical wave $ct = \bar{r}$ and $ct' = r'$ one obtains:

$$x'^0 = \gamma x^0 - \beta \gamma x^1$$
$$x'^1 = -\beta \gamma x^0 + \gamma x^1$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

$$\beta = \frac{v}{c} \leq 1$$
$$\gamma = (1-\beta^2)^{-1/2}$$

Then

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Some people define:

$$\left.\begin{aligned} \gamma &= \cosh\rho \\ \beta\gamma &= \sinh\rho \\ \tanh\rho &= \beta = v/c \end{aligned}\right\}$$

With this correspondence the Lorentz transformation look like a rotation about an imaginary angle $i\rho$.

In the x^1, x^2, x^3, x^4 notation the transformation matrix is given by: $\bar{v} = v\hat{k}$

$$M^a{}_b = \frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$$

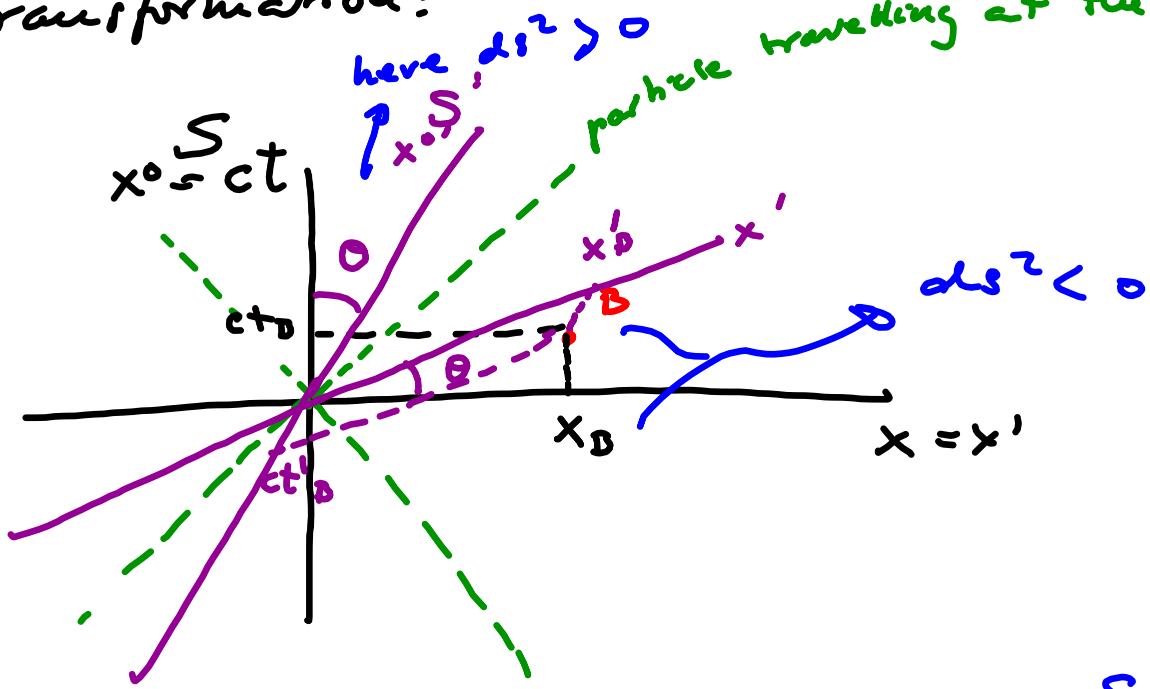
Here you assign:

$$\gamma = \cos\theta$$

$$i\beta\gamma = \sin\theta$$

$$\tan\theta = i\frac{v}{c}$$

Graphic representation of the Lorentz transformation:



$$ds^2 = c^2t^2 - r^2$$

S': $\beta = 0.6$

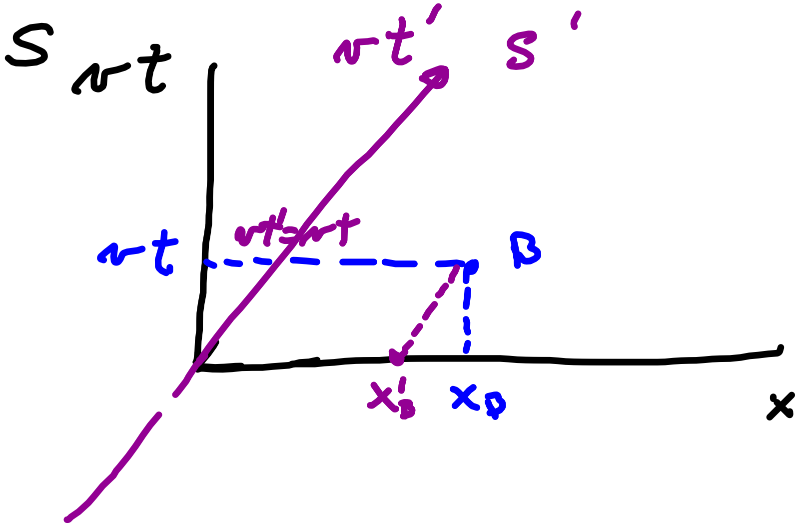
$\gamma = 1.25$

$\tan^{-1}(0.6) = 31°$

In a Galilean transformation only the t' axis moves:

$$t' = t \qquad \tan\theta = v$$

$$x' = x - vt$$



Levi-Civita tensor in Minkowski space:

$$\varepsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if } \alpha\beta\gamma\delta \text{ are cyclic.} \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ are non-cyclic} \\ 0 & \text{if any index is repeated.} \end{cases}$$

$$\varepsilon_{\alpha\beta\gamma\delta} = g_{\alpha\rho}\, g_{\beta\tau}\, g_{\gamma\sigma}\, g_{\delta\omega}\, \varepsilon^{\rho\tau\sigma\omega}$$

if $\alpha\beta\gamma\delta = 0123$ $\quad \varepsilon^{0123} = 1$

$$\varepsilon_{0123} = \underbrace{g_{00}}_{1}\, \underbrace{g_{11}}_{-1}\, \underbrace{g_{22}}_{-1}\, \underbrace{g_{33}}_{-1}\, \underbrace{\varepsilon^{0123}}_{1} = -1$$

Different than in 3D space.

4 - divergence:

$$\partial_\alpha A^\alpha = \partial_0 A^0 + \bar{\nabla} \cdot \bar{A} \qquad \text{scalar}$$

$\partial \quad (\partial_0, \bar{\nabla})$

$\partial \quad (A^0, \bar{A})$