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Tensors in Minkowski space.

$A^\mu = (A^0, A^1, A^2, A^3)$ is a 4 vector.

4-divergence:

$$\partial_\alpha A^\alpha = \partial_0 A^0 + \bar{\nabla} \cdot \bar{A} \qquad \text{scalar.}$$

$(A^0, \bar{A})$

$(\partial_0, \bar{\nabla})$

Invariants: tensors of rank 0.

Consider:

$$A^{\mu} = (dx^0, 0, 0, 0)$$

$$B^{\nu} = (0, dx^1, 0, 0)$$

$$C^{\rho} = (0, 0, dx^2, 0)$$

$$D^{\tau} = (0, 0, 0, dx^3)$$

$$H^{\mu\nu\rho\tau} = A^{\mu} B^{\nu} C^{\rho} D^{\tau}$$ tensor of rank 4.

It has $4^4 = 256$ elements but only 1 is non-zero.

We can define a pseudoscalar in terms of $H^{\mu\nu\rho\tau}$:

$$V = \varepsilon_{\mu\nu\rho\tau} H^{\mu\nu\rho\tau} = dx^0 dx^1 dx^2 dx^3 = d^4x$$

$V = d^4x$ is the volume element in 4D and it is a pseudoscalar $\therefore$ is invariant under a transformation from S to S'!

Notice that

$dx^1 dx^2 dx^3$ is NOT invariant

Physical quantities in 4-space:

Continuity equations:

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \bar{J} = 0 \quad (1) \qquad \bar{J} = \rho \bar{v} \quad \text{3-vector}$$

$\rho$ : scalar

In 4-space (1) works only if we define a 4-vector current as:

$$(2) \quad J^\mu = (J^0, \bar{J}) = (c\rho, \bar{J}) \quad \text{4-vector current}$$

How do we write ① in terms of ② ?

$$\frac{\partial \rho}{\partial t} = \frac{c}{c}\frac{\partial \rho}{\partial t} = \frac{\partial (c\rho)}{\partial (ct)} = \frac{\partial J^0}{\partial x^0}$$

$$\bar{\nabla} \cdot \bar{J} = \partial_i J^i \quad \text{with } i = 1,2,3$$

In the book (SI system):

$$j^\mu = \left(\rho, \frac{\bar{J}}{c}\right)$$

Now

$$\partial_\mu J^\mu = 0 \quad \text{is eq. ①}$$

Since we get

$$\partial_\mu J^\mu = \partial_0 J^0 + \partial_i J^i \equiv \frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \bar{J} = 0 .$$

Another 4-vector is:

$$A^\mu = (\varphi, \bar{A})$$

$\bar{A}$ : vector potential.

$\varphi$ : scalar potential.

$$\bar{B} = \bar{\nabla} \times \bar{A} \quad (1)$$

$$\bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi \quad (2)$$

In the book:

$$A^\mu = \varepsilon_0(\varphi, c\bar{A})$$

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\varphi$$

SI system.

In classical e+m we know that $\bar{A}$ and $\varphi$ satisfy wave-equations:

$$\left.\begin{aligned} \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi &= 4\pi\rho \\ \frac{1}{c^2}\frac{\partial^2\bar{A}}{\partial t^2} - \nabla^2\bar{A} &= \frac{4\pi}{c}\bar{J} \end{aligned}\right\} \quad (3)$$

Now using $J^\nu = (\varphi, \bar{J})$ and $A^\nu = (\varphi, \bar{A})$ we can write (3) in a compact form:

$$\boxed{\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu}$$

Ex: $\nu = 0$ $\quad \partial_0\partial^0 = \frac{\partial^2}{\partial(ct)^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

$\partial_\mu\partial^\mu A^0 = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{4\pi}{\cancel{c}}\cancel{c}\rho$

$\partial_i\partial^i = -\frac{\partial^2}{\partial x^{i\,2}}$ $\quad i = 1,2,3$

Lorentz gauge:

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \bar{\nabla}\cdot\hat{A} = 0$$

$$\downarrow$$

$$\partial_\alpha A^\alpha = 0$$

(looks like the Coulomb gauge in 3D space).

In electrostatic you use $\bar{\nabla}\cdot\hat{A} = 0$ (Coulomb gauge)

Since $\bar{\nabla}\times\hat{A} = \bar{B} \Rightarrow$

$\bar{\nabla}\cdot\hat{A}$ can be arbitrarily chosen.

It's value is called a "gauge".

Now we will define a rank 2 tensor in 4-space:

$$F^{\mu\lambda} = \partial^{\mu}A^{\lambda} - \partial^{\lambda}A^{\mu} = -F^{\lambda\mu}$$ (4)

antisymmetric by definition.

All diagonal elements $F^{\lambda\lambda} = 0$.

We'll see that $F^{\mu\nu}$ can be expressed in terms of the components of $\bar{B}$ and $\bar{E}$.

Let's use (4) to calculate $F^{\mu\nu}$:

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} - \left(-\frac{\partial \varphi}{\partial x}\right) =$$

$$= \frac{\partial \varphi}{\partial x} + \frac{1}{c}\frac{\partial A_x}{\partial t} = -E_x$$ (Eq. 2 in previous pages)

Doing the same for all components we get:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

To write Maxwell's equations in a compact way let's define $\mathcal{F}^{\alpha\beta}$ the dual tensor of $F^{\mu\nu}$:

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\,\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} = \frac{1}{2}\,\varepsilon^{\alpha\beta\mu\nu} g_{\mu\gamma}\, g_{\nu\delta}\, F^{\gamma\delta}$$

Notice that we can construct

$$F_{\mu\nu} = g_{\mu\gamma}\, g_{\nu\delta}\, F^{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Sign of $E_i$ gets exchanged.

$$F_{01} = \overbrace{g_{00}}^{1}\, \overbrace{g_{11}}^{-1}\, F^{01} = -F^{01}$$

only possible value

$$= -(-E_x) = E_x$$

Now let's obtain $\mathcal{F}^{\alpha\beta}$:

$$\mathcal{F}^{01} = \frac{1}{2}\varepsilon^{01\gamma\delta}F_{\gamma\delta} = \frac{1}{2}\underbrace{\varepsilon^{0123}}_{1}\underbrace{F_{23}}_{-B_x} + \frac{1}{2}\underbrace{\varepsilon^{0132}}_{-1}\underbrace{F_{32}}_{B_x} = -B_x$$

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

pseudotensor of rank 2.

Now we can write:

$$\left.\begin{aligned} &\bar{\nabla}\cdot\bar{E} = 4\pi\rho \\ &\bar{\nabla}\times\bar{B} - \frac{1}{c}\frac{\partial\bar{E}}{\partial t} = \frac{4\pi}{c}\bar{J} \end{aligned}\right\} \quad \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

$$\left.\begin{aligned} &\bar{\nabla}\cdot\bar{B} = 0 \\ &\bar{\nabla}\times\bar{E} + \frac{1}{c}\frac{\partial\bar{B}}{\partial t} = 0 \end{aligned}\right\} \quad \partial_\alpha \mathcal{F}'^{\alpha\beta} = 0$$

Notice that $\bar{\nabla} \cdot \bar{B} = 0$ can be written as:

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

$\alpha = 1 \quad \beta = 2 \quad \gamma = 3$

# Differential Equations

For review on ordinary diff. eqs. see Ch. 7.

We will start on Ch. 9 with partial diff. eqs.

In Physics there are many differential eqs that need to be solved:

$\nabla^2 \psi = 0$ Laplace equation (homogeneous).

$\bar{F} = m\ddot{x}$ Newton

$\nabla^2 \psi = -\frac{\rho}{\varepsilon_0}$ Poisson (inhomogeneous).

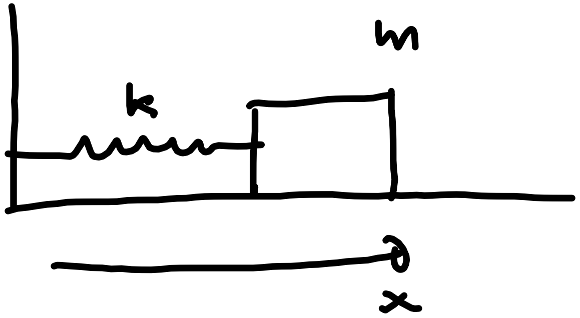
$$\nabla^2 \psi + k^2 \psi = 0$$ wave eq.

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = i\hbar \frac{\partial \psi}{\partial t}$$ Schrödinger's eq.

Techniques:

- Separation of variables
  For homogeneous problems that can be separated into several ordinary diff eqs..
- Green functions (inhomogeneous eqs.)
- Frobenius method (certain ordinary diff. eqs.)

In general you need boundary conditions to solve specific problems:




$$m\ddot{x} + kx = 0 \qquad (1)$$

$$x = A\cos(\omega t + \varphi) \qquad (2)$$

Plugging (2) in (1) you find that (2) solves (1)

if $\omega = \sqrt{\frac{k}{m}}$

Second order equation (contains second derivatives)

We need as many (order of the eq.) initial or boundary conditions as the

A and $\varphi$ are determined from $x_0$ and $v_0$ (initial position and velocity).

$$\left.\begin{aligned} x_0 &= A\cos\varphi \\ \dot{x}_0 &= v_0 = -A\omega\sin\varphi \end{aligned}\right\} \quad \begin{aligned} \varphi &= \tan^{-1}\left(\frac{-v_0}{x_0\omega}\right) \\ A &= \frac{x_0}{\cos\varphi} \end{aligned}$$