Divergence Theorem

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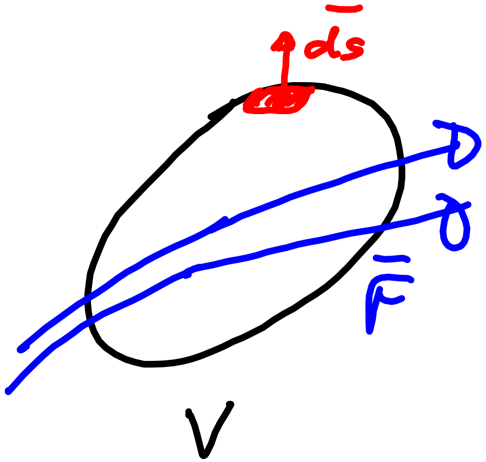
$$\int_V \bar{\nabla}\cdot\bar{F}\, dV = \underbrace{\oint \bar{F}\cdot d\bar{S}}_{\text{flux of } \bar{F} \text{ through } S.}$$

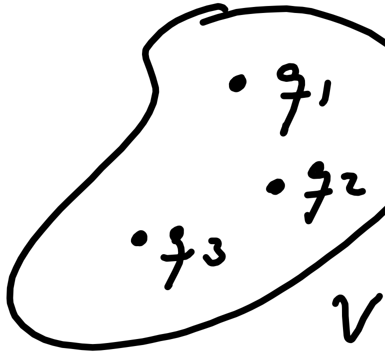


$d\bar{s}$ is a vector that is perpendicular to ds (surface) and points outside the volume.

Example:

1- Gauss' law:

$$\oint_S \bar{E}\cdot d\bar{s} = \frac{q_{encl}}{\varepsilon_0}$$



$$\oint_S \bar{E}\cdot d\bar{s} \underset{\text{div. theorem}}{=} \int_V \underbrace{\bar{\nabla}\cdot\bar{E}}_{\rho/\varepsilon_0}\, dV = \frac{1}{\varepsilon_0}\underbrace{\int_V \rho\, dV}_{\text{charge enclosed}} = \frac{q_{enc.}}{\varepsilon_0}$$

Maxwell's equation

ρ: density of charge

2) Poisson's equation:

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Maxwell's eq.)}$$

and $\bar{E} = -\bar{\nabla} V$ (electrostatics).

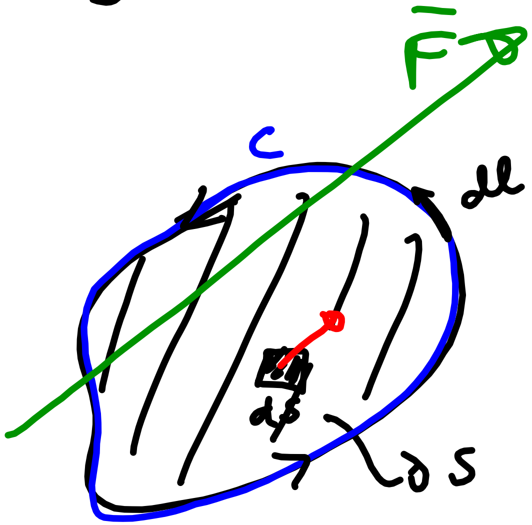
$$\bar{\nabla} \cdot \bar{E} = -\bar{\nabla}(\bar{\nabla} V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

Then $\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$ Poisson's eq. (inhomogeneous diff. eq.).

if $\rho = 0$ $\nabla^2 V = 0$ homogeneous diff. eq.
Laplace's eq.

Stoke's theorem

$$\int \bar{\nabla} \times \bar{F} \cdot d\bar{S} = \oint_c \bar{F} \cdot d\bar{l}$$

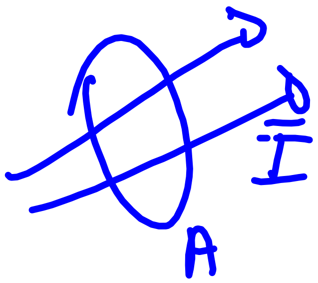


Ex: Ampère's law:

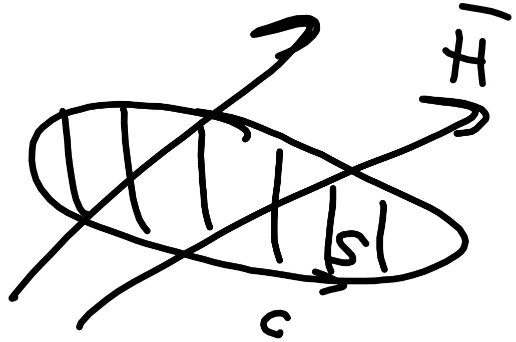
$$= \int_S \underbrace{\bar{\nabla} \times \bar{H}}_{\bar{J}} \cdot \bar{dS} = \int_S \bar{J} \cdot \bar{dS} = \bar{I}_{\text{through surface}}$$

$\hookrightarrow \oint_c \bar{H} \cdot \bar{dl}$ circulation of $\bar{H}$ (stoke's theorem)

$\bar{J}$: density of current throug an area A:

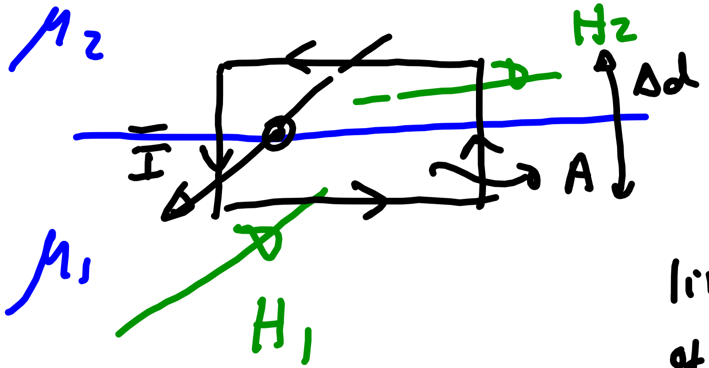


$$\bar{J} = \frac{\bar{I}}{A}$$



HW#3: You will use Stoke's theorem to find the boundary conditions for the tangential components of $\bar{H}$ at the boundary between two materials.

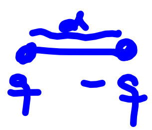
$$\bar{J} = \frac{\bar{I}}{A}$$



$$\bar{K} = \lim_{\substack{\Delta d \to 0 \\ J \to \infty}} \bar{J}\,\Delta d$$

linear density of current

Similar to definition of point-like dipole:



$p = q d$

point-like $\bar{p} = \lim_{\substack{q \to \infty \\ d \to 0}} q d$

Homework: You'll have to find the dual basis for the oblique system and show that

$\bar{r} = x'_i e^{i'}$

you will see that in dual system x'_i are the parallel projections.

$e^{i'}$ are not unit vectors - Find them using that

$e'_i e^{'j} = \delta_i^{\ j}$

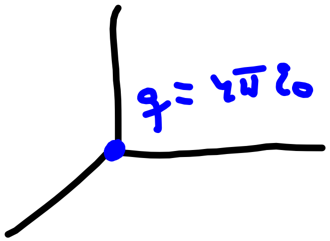
- Find e'_i in orthogonal basis.
- Then get $e^{'i}$ in orthogonal basis.

Dirac Delta "Function"

Ex: If $V = V(r)$ $\quad r = \sqrt{x^2+y^2+z^2} = (x_i x^i)^{1/2}$

$\bar{\nabla} V = \frac{\partial V}{\partial r}\hat{r}$ gradient is perpendicular to the equipotential spheres.

Consider a point charge $q = 4\pi\varepsilon_0$ centered at the origin:



$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$ $\quad \bar{E} = -\bar{\nabla} V$

$\bar{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \equiv \frac{\hat{r}}{r^2}$ $\quad V = \frac{q}{4\pi\varepsilon_0 r} \equiv \frac{1}{r}$

Let's apply Gauss' theorem to this example:

$$\oint_S \bar{E}\cdot\bar{dS} = \int_V \bar{\nabla}\cdot\bar{E}\,dV = \int \frac{\rho(\bar{x})}{\varepsilon_0}\,dV = \frac{q_{encl}}{\varepsilon_0} = \begin{cases} 4\pi & \text{if } \xi \in V \\ 0 & \text{if } \xi \text{ outside } V \end{cases}$$

$\shortparallel$

$$\frac{\hat{r}}{r^2} = \frac{\bar{r}}{r^3}$$

$$\oint_S \frac{\hat{r}}{r^2}\cdot\bar{dS} = \int_V \bar{\nabla}\underbrace{\left(-\nabla\left(\frac{1}{r}\right)\right)}\,dV = -\int_V \nabla^2\left(\frac{1}{r}\right)dV$$

Notice that

$$-\nabla^2\left(\frac{1}{r}\right) = \bar{\nabla}\cdot\left(\frac{\bar{r}}{r^3}\right) = \frac{(-2+2)}{r^3} = \frac{0}{r^3} = \begin{cases} 0 & \text{if } r \neq 0 \\ \text{undefined} & \text{if } r = 0 \end{cases}$$

Last time we found that $\bar{\nabla}\cdot(\bar{r}\,r^{n-1}) = (n+2)\,r^{n-1}$

if $n=-2$ $\bar{r}\,r^{n-1} = \frac{\bar{r}}{r^3}$

You see that $\nabla^2\left(\frac{1}{r}\right)$ is not a well-defined function in all space. However,

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV$$ is well defined. Then

we define:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(r) = -4\pi\,\delta(x)\,\delta(y)\,\delta(z)$$

Since
$$\int \nabla^2\left(\frac{1}{r}\right) dV = \begin{cases} -4\pi & \text{if } V \text{ includes the origin.} \\ 0 & \text{if } V \text{ does not include the origin.} \end{cases}$$

Then we see that

$$\int_V \delta(r)\, dV = 1 \text{ if } \begin{cases} r=0 \text{ is in } V. \\ 0 \text{ otherwise.} \end{cases}$$

Properties of $\delta(x)$:

$\delta(x)$ is NOT a function.

$\delta(x) = 0$ if $x \neq 0$

$$f(0) = \int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx$$

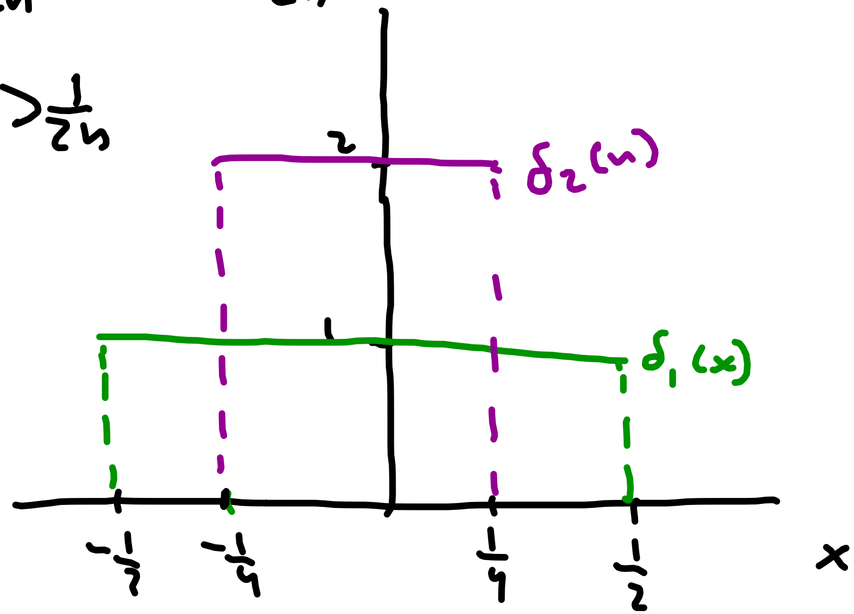
if $f(x) = 1 \Rightarrow \int_{-\infty}^{\infty} \delta(x)\, dx = f(0) = 1$

$\delta(x)$ is obtained as the limit of a sequence of functions:

Ex:

$$\delta_n(x) = \begin{cases} 0 & \text{if } x < -\frac{1}{2n} \\ n & \text{if } -\frac{1}{2n} \leq x \leq \frac{1}{2n} \\ 0 & \text{if } x > \frac{1}{2n} \end{cases}$$

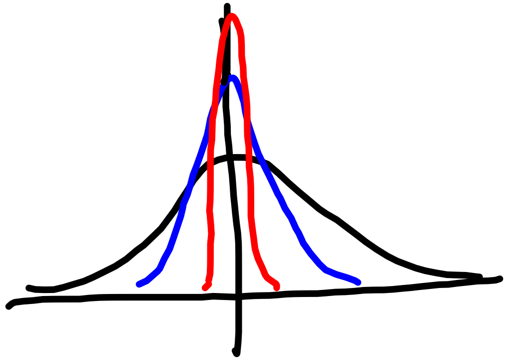
support functions.



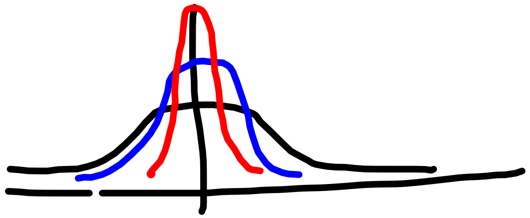
You "see" that the limit for $n \to \infty$ is $\delta(x)$.

Other examples of support functions: (Ch. 1)

- $\delta_n(x) = \dfrac{n}{\sqrt{\pi}} e^{-n^2x^2}$ gaussians

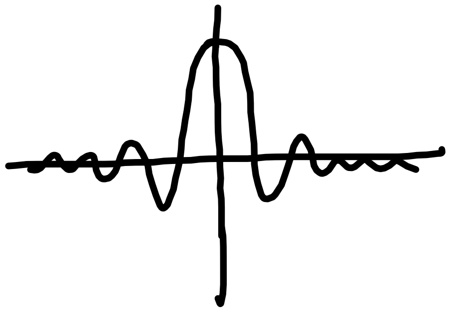


- $\delta_n(x) = \dfrac{n}{\pi} \dfrac{1}{1+n^2x^2}$



Lorentzians.

- $\delta_n(x) = \dfrac{\sin nx}{\pi x} = \dfrac{1}{2\pi}\displaystyle\int_{-n}^{n} e^{ixt}\,dt$



As we said

$$\lim_{n\to\infty} \delta_n(x)$$ does not exist.

(it is NOT a function).

but

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)\, f(x)\, dx \equiv f(0)$$

if $\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1$ the functions $\delta_n(x)$ are normalized.

Now we can define $\delta(x)$ by:

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \underbrace{\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x)\, dx}_{\text{this is the way in which } \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0) \text{ is demonstrated.}} = f(0)$$

Example:

$$\lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x)\, dx = \lim_{n\to\infty} n \underbrace{\int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x)\, dx}_{f(0)\Delta = f(0)\frac{1}{n}} =$$

$$= \lim_{n\to\infty} n \frac{f(0)}{n} = f(0)$$

0 : is the middle point of the interval

Properties of $\delta(x)$:

- $$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \int f(y+a)\,\delta(y)\,dy = f(a)$$

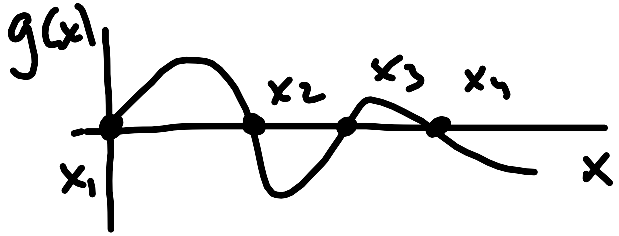
$y = x-a$

$dy = dx$

- $$\int_{-\infty}^{\infty} f(x)\,\delta(g(x))\,dx = \int_{-\infty}^{\infty} f(x) \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,dx$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$$ where $g(x_i)=0$

$g'(x_i) = \frac{\partial g}{\partial x}\Big|_{x=x_i}$ if $g'(x_i)\neq 0$.



Example:

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{\infty} f(x)\,\frac{\delta(x-0)}{|a|}\,dx = \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

$$= \frac{1}{|a|} f(0)$$

$g(x) = ax$

$x_1 = 0$

$g'(x) = a$

$g'(0) = a$

- $$\int_{-\infty}^{\infty} f(x)\,\delta'(x-x')\,dx = \underbrace{f(x)\,\delta(x-x')\Big|_{-\infty}^{\infty}}_{0 \text{ då } \delta(x)=0 \text{ för } \pm\infty.} - $$

$$- \int_{-\infty}^{\infty} f'(x)\,\delta(x-x')\,dx =$$

$$= \boxed{-f'(x')}$$