9/16

Dirac delta function in 3D:

$$\delta(\bar{x}-\bar{X}) = \delta(x_1 - X_1)\,\delta(x_2 - X_2)\,\delta(x_3 - X_3)$$

$$\int_V \delta(\bar{x}-\bar{X})\, d^3x = \begin{cases} 1 \text{ if } \bar{X} \text{ is in } V. \\ 0 \text{ if } \bar{X} \text{ is not in } V. \end{cases}$$

Examples: $\rho(\bar{x}) = q\,\delta(\bar{r}-\bar{x})$ for a point charge.

we see that

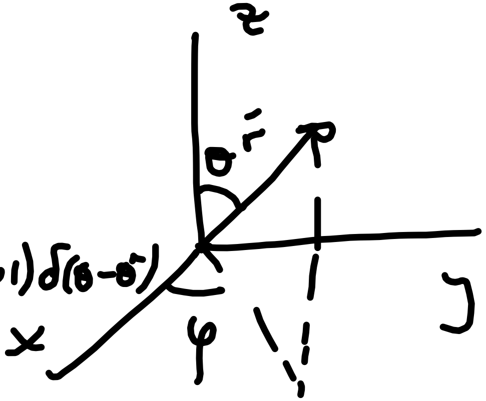
$$\int_V \rho(\bar{x})\, d^3x = q \int_V \delta(\bar{r}-\bar{x})\, d^3x = q \text{ if } V = \text{all space.}$$

Since many times boundary conditions are given on spheres or cylinders rather than planes we need to express $\delta(\bar{x}-\bar{x}')$ in different systems of coordinates:

Spherical coordinates: (r, θ, φ)

We know that $\int_{\text{all space}} \delta(\bar{r})\, dV = 1$

$$1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} A \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\varphi}_{dV} \, \delta(r-r')\,\delta(\varphi-\varphi')\,\delta(\theta-\theta')$$



r

$$= \underbrace{\int_0^\infty A\,\delta(r-r')\,r^2\,dr}_{A r'^2} \underbrace{\int_0^\pi \delta(\theta-\theta')\sin\theta\,d\theta}_{\sin\theta'} \underbrace{\int_0^{2\pi} \delta(\varphi-\varphi')\,d\varphi}_{1}$$

Then $A = \dfrac{1}{r'^2 \sin\theta'}$

Then

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{r'^2 \sin\theta'} = \frac{\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')}{r^2\sin\theta}$$

In general:

$$\delta(\bar{x} - \bar{x}') = \frac{\delta(\xi_1 - \xi_1')\,\delta(\xi_2 - \xi_2')\,\delta(\xi_3 - \xi_3')}{|J(x_i, \xi_i)|}$$

Jacobian

$$J(x_i, \xi_i) = \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{vmatrix} \qquad x_i = x_i(\xi_j)$$

If $\{\xi^i\} = (r, \theta, \varphi)$

$$x = r \operatorname{sen}\theta \cos\varphi$$
$$y = r \operatorname{sen}\theta \operatorname{sen}\varphi$$
$$z = r \cos\theta$$

$$J = \begin{vmatrix} \operatorname{sen}\theta \cos\varphi & r\cos\theta\cos\varphi & -r\operatorname{sen}\theta\operatorname{sen}\varphi \\ \operatorname{sen}\theta\operatorname{sen}\varphi & r\cos\theta\operatorname{sen}\varphi & +r\operatorname{sen}\theta\cos\varphi \\ \cos\theta & -r\operatorname{sen}\theta & 0 \end{vmatrix} = r^2 \operatorname{sen}\theta$$

If $\{\xi_i\} = (r, \cos\theta, \varphi)$

$$\delta(\bar{r} - \bar{r}') = \frac{\delta(r-r')\,\delta(\cos\theta - \cos\theta')\,\delta(\varphi - \varphi')}{r^2}$$

Expansion of $\delta(x-x')$ in terms of orthogonal functions. [Review: Ch. 5.1.6]

$\{\varphi_n(x)\}$ are orthogonal in the interval (a,b):

$$\int_a^b \varphi_m^*(x)\,\varphi_n(x)\,dx = \delta_{m,n}\,C$$

C: constant
if $C = 1$
$\{\varphi_n(x)\}$: orthonormal

Any "good behaved" function $f(x)$ can be expanded in terms of $\{\varphi_n(x)\}$:

$$f(x) = \sum_{n=0}^{\infty} b_n \varphi_n(x)$$

with

$$b_n = \int_a^b f(x)\, \varphi_n^*(x)\, dx$$

D/

$$\int_a^b \varphi_m^*(x) f(x)\, dx = \overbrace{\sum_{n=0}^{\infty} b_n \underbrace{\int_a^b \varphi_m^*(x)\, \varphi_n(x)\, dx}_{\delta_{m,n}}}^{b_m}$$

Now let's do the same for $\delta(x-t)$:

$$\delta(x-t) = \sum_{m=0}^{\infty} a_m(t)\, \varphi_m(x) \quad \text{i}$$

Let's find $a_m(t)$:

$$a_m(t) = \underbrace{\int_a^b \varphi_m^*(x)\, \delta(x-t)\, dx}_{\varphi_m^*(t)} \quad \text{i} \quad =$$

$$= \int_a^b \varphi_m^*(x) \sum_{m=0}^{\infty} a_m(t)\, \varphi_m(x)\, dx =$$

$$= \sum_{m=0}^{\infty} a_m(t) \underbrace{\int_a^b \varphi_n^*(x)\,\varphi_m(x)\,dx}_{\delta_{n,m}} =$$

$$= a_n(t)$$

Then $a_n(t) = \varphi_n^*(t)$

$$\therefore \boxed{\delta(x-t) = \sum_{m=0}^{\infty} \varphi_m^*(t)\,\varphi_m(x)}$$

Example: $\left\{ \varphi_n(x) \right\} = \left\{ \sqrt{\frac{2}{a}} \cos \frac{2n\pi x}{a} \right\}$ in $\left(-\frac{a}{2}, \frac{a}{2}\right)$

orthonormal since

$$\frac{2}{a} \int_{-a/2}^{a/2} \cos \frac{2n\pi x}{a} \cos \frac{2m\pi x}{a} \, dx = \delta_{m,n}$$

Then:

$$\delta(x-t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{2\pi n x}{a}$$

Multiply both sides by $\sqrt{\frac{2}{a}} \cos \frac{2\pi m x}{a}$ and integrate over x on $\left(-\frac{a}{2}, \frac{a}{2}\right)$ to find $a_n(t)$:

$$\underbrace{\sqrt{\frac{2}{a}}\int_{-a/2}^{a/2} \delta(x-t)\cos\frac{2\pi m x}{a}\,dx}_{\cos\frac{2\pi m t}{a}} = \sum_{m=0}^{\infty} a_m(t)\int_{-\frac{a}{2}}^{a/2}\cos\frac{2\pi m x}{a}\cdot$$

$$\underbrace{\frac{2}{a}\cdot\cos\frac{2\pi m x}{a}\,dx}_{1}$$

$$a_m(t) = \sqrt{\frac{2}{a}}\cos\frac{2m\pi t}{a}$$

$$\boxed{\delta(x,t) = \frac{2}{a}\sum_{m=0}^{\infty}\cos\frac{2\pi m t}{a}\cos\frac{2m\pi x}{a}}$$

Tensor analysis (Ch. 4)

A tensor of rank k in dimension N has N^k elements.

$k=0 \Rightarrow$ scalar 1 component

$k=1 \Rightarrow$ vector N components

$\vdots$

So far we know that:

- Tensors of rank 0 are invariant under transformations.

- Tensors of rank 1 transform according to:

$$A^{i'} = \frac{\partial x'^{i}}{\partial x^{j}} A^{j}$$ for contravariant components.

$$B_{j}' = \frac{\partial x^{i}}{\partial x'^{j}} B_{i}$$ for covariant components.

x'^{i} : components in S'

x^{i} : components in S

We use Einstein's notation summing over repeated indices.

Now let's study tensors with $k > 1$:

Direct product or outer product of tensors:

$$A^{\mu} B^{\nu} = C^{\mu\nu}$$

$C^{\mu\nu}$ has rank 2 and it results from the direct product of A^{μ} with B^{ν}.

In 2D:

$$A^{\mu} = (A^1, A^2) \qquad B^{\nu} = (B^1, B^2)$$

$$C^{\mu\nu} = \begin{pmatrix} A^1 B^1 & A^1 B^2 \\ A^2 B^1 & A^2 B^2 \end{pmatrix} = \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix}$$

How does $C^{\mu\nu}$ transform as we go from S to S'?

$$C'^{\mu\nu} = A'^{\mu} B'^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x'^{\nu}}{\partial x^{\beta}} B^{\beta} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \underbrace{A^{\alpha} B^{\beta}}_{C^{\alpha\beta}} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} C^{\alpha\beta}$$

$C^{\alpha\beta}$ is our prototype contravariant tensor of rank 2.

Notice that I can form the outer product of covariant and contravariant vectors then I can define:

$$C^{\mu}{}_{\nu} = A^{\mu} B_{\nu}$$

$$C_{\mu}{}^{\nu} = A_{\mu} B^{\nu}$$

} mixed rank tensors

$$C_{\mu\nu} = A_{\mu} B_{\nu}$$

} covariant tensor of rank 2.

You can show that

$$C'^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} C^{\alpha}{}_{\beta}$$

$$C'_{\mu}{}^{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} C_{\alpha}{}^{\beta}$$

$$C'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} C_{\alpha\beta}$$

Higher rank tensors can be constructed by performing the outer product of several vectors or of tensors of lower rank:

$$C^{\mu\nu} B^{\sigma} = T^{\mu\nu\sigma}$$

contravariant tensor of rank 3.

The number of indices defines the rank of the tensor.

Ex: in 2D

$$T^{\mu\nu\sigma} = \begin{array}{cc|cc} T^{111} & T^{112} & T^{211} & T^{212} \\ T^{121} & T^{122} & T^{221} & T^{222} \end{array}$$

(2×2×2 cube)

8 components.

How does $T^{\mu\nu\sigma}$ transform?

$$T'^{\mu\nu\sigma} = C'^{\mu\nu} B'^{\sigma} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} C^{\alpha\beta} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}} B^{\gamma} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x'^{\sigma}}{\partial x^{\gamma}} T^{\alpha\beta\gamma}$$

sums over α, β and γ

Other examples:

$$B^{\nu}{}_{\mu}{}^{\alpha} T_{\delta\varepsilon}{}^{\rho} = M^{\nu}{}_{\mu}{}^{\alpha}{}_{\delta\varepsilon}{}^{\rho}$$ mixed tensor.

$$M'^{\nu}{}_{\mu}{}^{\alpha}{}_{\delta\varepsilon}{}^{\rho} = \frac{\partial x'^{\nu}}{\partial x^{a}} \frac{\partial x^{b}}{\partial x'^{\mu}} \frac{\partial x'^{\alpha}}{\partial x^{c}} \frac{\partial x^{d}}{\partial x'_{\delta}} \frac{\partial x^{e}}{\partial x'_{\varepsilon}} \frac{\partial x'^{\rho}}{\partial x_{f}} M^{a}{}_{b}{}^{c}{}_{de}{}^{f}$$