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Addition of tensors:

Two tensors can be added if they have the same rank and their indices transform in the same way:

$$C^{ab} = A^{ab} + B^{ab}$$

or

$$A^{a}{}_{b} + B^{a}{}_{b} = C^{a}{}_{b}$$

But

$$\cancel{A^{a}{}_{b} + B^{ab}} \quad \text{is wrong!}$$

Contraction of tensors

If two indices of a tensor of rank n (one is covariant and the other contravariant) are repeated implying a sum over the indicated components the 2 indices are contracted and the result is a tensor of rank $n-2$.

Ex: $N=2$

$$C^{\mu}{}_{\nu} = \begin{pmatrix} C^1{}_1 & C^1{}_2 \\ C^2{}_1 & C^2{}_2 \end{pmatrix} = \mathbb{C}$$

Let's contract the two indices:

$$C^{\mu}{}_{\mu} = C^1{}_1 + C^2{}_2 \qquad \text{scalar.}$$

$C^{\mu}{}_{\mu}$ = trace of $\mathbb{C}$ - Tensor of rank 0.

Notice:

$$C^{\mu\nu} = \begin{pmatrix} c^{11} & c^{12} \\ c^{21} & c^{22} \end{pmatrix}$$

As a matrix I can calculate the trace of $C^{\mu\nu}$ but $c^{11} + c^{22}$ is an scalar <u>but</u> if is NOT a tensor.

Example:

Let's $C^{\mu\nu} = x^\mu x^\nu = \begin{pmatrix} x^1 x^1 & x^1 x^2 \\ x^2 x^1 & x^2 x^2 \end{pmatrix}$ in $N=2$.

In cartesian coordinates $\mathrm{tr}(C^{\mu\nu}) = x^{1\,2} + x^{2\,2} = r^2$.

Let S' be the oblique system:

$$C'^{\mu\nu} = \begin{pmatrix} x'^1 x'^1 & x'^1 x'^2 \\ x'^2 x'^1 & x'^2 x'^2 \end{pmatrix}$$

$$\mathrm{tr}\left(C'^{\mu\nu}\right) = \left(x'^1\right)^2 + \left(x'^2\right)^2 \neq r^2$$

In S' the scalar invariant is obtained from $C'_\mu{}^\nu$:

$$C'_\mu{}^\nu = \begin{pmatrix} x'_1 x'^1 & x'_1 x'^2 \\ x'_2 x'^1 & x'_2 x'^2 \end{pmatrix}$$

Now $C'_\mu{}^\mu = x'_1 x'^1 + x'_2 x'^2 =$
$= r^2$

This shows that in general a scalar may not be a tensor of rank 0.

Every tensor of rank 0 is an scalar but all scalars are <u>not</u> tensors of rank 0.

Let's see that $C'^{\mu}{}_{\mu}$ is an scalar:

$$C'^{\mu}{}_{\mu} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}}}_{\frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \frac{\partial x^{\beta}}{\partial x^{\alpha}} = \delta^{\beta}{}_{\alpha}} C^{\alpha}{}_{\beta} = \delta^{\beta}{}_{\alpha} C^{\alpha}{}_{\beta} = C^{\alpha}{}_{\alpha}$$

transforms as tensor of rank 0.

In general consider

$$A^i{}_j \, B^{kl} = T^i{}_j{}^{kl}$$

rank 4

I can do the following contractions:

- $A^i{}_i \, B^{kl} = T^i{}_i{}^{kl} = N^{kl}$
- $A^i{}_j \, B^{jl} = T^i{}_j{}^{jl} = N^{il}$
- $A^i{}_j \, B^{kj} = T^i{}_j{}^{kj} = N^{ik}$

rank 2 after contraction.

you cannot contract: k, l or ik or il.

Matrix Multiplication vs tensor contraction

All tensors of rank 2 are matrices but not all matrices are tensors of rank 2.

Consider A^{ij} and B^{kl} (2 tensors of rank 2).

As matrices we can define

$$C^{ij} = \sum_{k=1}^{N} A^{ik} B^{kj}$$

is C^{ij} a tensor?

A contravariant tensor of rank 2 transforms like

$$T'^{ij} = \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} T^{\alpha\beta}$$

Let's consider C^{ij}:

$$C'^{ij} = A'^{ik} B'^{kj} = \frac{\partial x'^i}{\partial x^a} \frac{\partial x'^k}{\partial x^b} A^{ab}$$

$$\frac{\partial x'^k}{\partial x^c} \frac{\partial x'^j}{\partial x^d} B^{cd} \neq \frac{\partial x'^i}{\partial x^a} \frac{\partial x'^j}{\partial x^b} C^{ab}$$

$\therefore$ C^{ij} is not a tensor.

In order to obtain a tensor from the matrix multiplication of A and B you need to use a mixed form of one or both of them:

$$C^{ij} = A^i{}_k B^{kj}$$

$$C'^{ij} = \frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^k} A^\alpha{}_\beta \frac{\partial x'^k}{\partial x^\gamma}\frac{\partial x'^j}{\partial x^\delta} B^{\gamma\delta} =$$

$$= \frac{\partial x'^i}{\partial x^\alpha}\underbrace{\frac{\partial x^\beta}{\partial x'^k}\frac{\partial x'^k}{\partial x^\gamma}}_{\partial x^\beta/\partial x^\gamma = \delta^\beta{}_\gamma}\frac{\partial x'^j}{\partial x^\delta} A^\alpha{}_\beta B^{\gamma\delta} =$$

$$= \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\delta} \delta^\beta{}_\gamma \, A^\alpha{}_\beta \, B^{\gamma\delta} =$$

$$= \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\delta} \underbrace{A^\alpha{}_\beta \, B^{\beta\delta}}_{C^{\alpha\delta}} =$$

$$= \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\delta} C^{\alpha\delta}$$

transforms as a contravariant tensor of rank 2.

Tensor contractions vs similarity transformations.

A matrix in a basis S is transformed to a basis S' by doing

$$A' = U A U^{-1} \qquad (1)$$

where U is the change of basis matrix.

Now assume that S' is a system rotated with respect to S and that A is a tensor of rank 2.

In this case $U \equiv M^i{}_j = \dfrac{\partial x'^i}{\partial x^j}$

and $U^{-1} \equiv A^i{}_j = \dfrac{\partial x^i}{\partial x'^j}$

If A is a tensor:

$$A'^i{}_l = \frac{\partial x'^i}{\partial x^j} \frac{\partial x^m}{\partial x'^l} A^j{}_m =$$

$$= \frac{\partial x'^i}{\partial x^j} A^j{}_m \frac{\partial x^m}{\partial x'^l} \Rightarrow A' = U A U^{-1}$$

similarity transformation.

but notice that

$$A'^{il} = \underbrace{\frac{\partial x'^i}{\partial x^j}}_{U} \underbrace{\frac{\partial x'^l}{\partial x^m}}_{U} A^{jm}$$

this is NOT a similarity transformation.

Kronecker Delta

Is it a tensor?

We know that

$$\delta_{k\ell} = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{otherwise} \end{cases}$$

in 2D

$$\delta_{k\ell} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It could be a tensor of rank 2. To prove it we need to see how it transforms from S to S'!

since $x'^i = x'^i(x^j)$

$$\delta'_{ij} = \frac{\partial x'^i}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}\delta_{kl} =$$

then δ_{ij} transforms like $\delta^i{}_j$ $\therefore$ it is a mixed tensor of rank 2:

$$\boxed{\delta'^i{}_j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}\delta^l{}_k}$$

Tensor symmetry properties.

In general

$$A^{mn} \neq A^{nm}$$

general tensors are not symmetric.

However, if for a tensor B^{mn}

$$B^{mn} = B^{nm}$$

then B^{mn} is a symmetric tensor.

Some tensor satisfied that

$$C^{ab} = -C^{ba}$$

then C^{ab} is an anti-symmetric tensor.

Notice that $C^{aa} = 0$ since

$C^{aa} = -C^{aa}$ for an antisymmetric tensor,

then $C^{a}{}_{a} = 0$ for an antisymmetric tensor

trace.

Independent components.

A tensor of rank k in dimension N has N^k components. However, if the tensor is symmetric or anti-symmetric for some pair of indices its number of independent components is reduced.

If a tensor of rank 2 is symmetric:

$$
\begin{matrix}
a_{11} & a_{12} & \cdots & a_{1N} \\
a_{12} & a_{22} & \cdots & a_{2N} \\
a_{13} & & \ddots & \vdots \\
\vdots & & & a_{NN} \\
a_{1N} & - & - & -
\end{matrix}
$$

a_{ij} : dependent

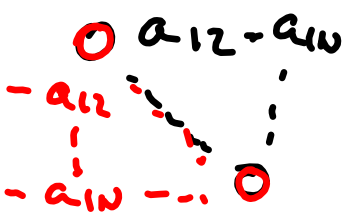
independent : $N + N-1 + N-2 + \cdots + 1$

$$= N^2 - \sum_{i=1}^{N-1} i = N^2 - \frac{N^2+N}{2} = \frac{N(N+1)}{2}$$

Thus, a symmetric tensor of rank 2 has $\frac{N(N+1)}{2}$ independent components:

N	total	independent
2	4	3
3	9	6
$\vdots$		
etc.		

For an antisymmetric tensor of rank 2:



diagonal elements

indep: $\dfrac{N(N+1)}{2} - N = \dfrac{N(N-1)}{2}$

For $N=2$ only 1 indep. element

For $N=3$ " 3 " "

Higher rank tensors

If T^{ijkl} symmetric under exchange of i and j.

So $T^{ijkl} = T^{jikl}$

In 3D: $3^4 = 81 = 3 \times 3 \times 3 \times 3$ for each index

But for ij we have 6 independent instead of 9

then we have $6 \times 3 \times 3 = 54$ independent components.