Example: Stress and Elasticity Tensors.

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Hook's law:

$$\bar{F} = k\,\bar{x}$$

In solids:

elasticity

$$\sigma = c\,\varepsilon$$

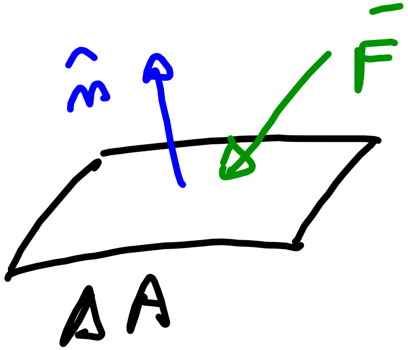
strain (deformation)

stress
(external forces)

In non-isotropic solids:

$$\sigma_{\alpha\beta} = \lim_{\Delta A \to 0} \frac{\Delta F_\alpha}{\Delta A_\beta}$$

stress tensor (rank 2)



Force per unit area.

Strain:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

u_α: displacement from equilibrium of a point in the solid.

C is now a tensor of rank 4:

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\,\varepsilon_{\gamma\delta}$$

(cartesian system assumed)

$C_{\alpha\beta\gamma\delta}$ is a rank 4 tensor in 3D:

$C_{\alpha\beta\gamma\delta}$ has $3^4 = 81$ components.

But $\varepsilon_{\gamma\delta}$ and $\sigma_{\alpha\beta}$ are symmetric tensors

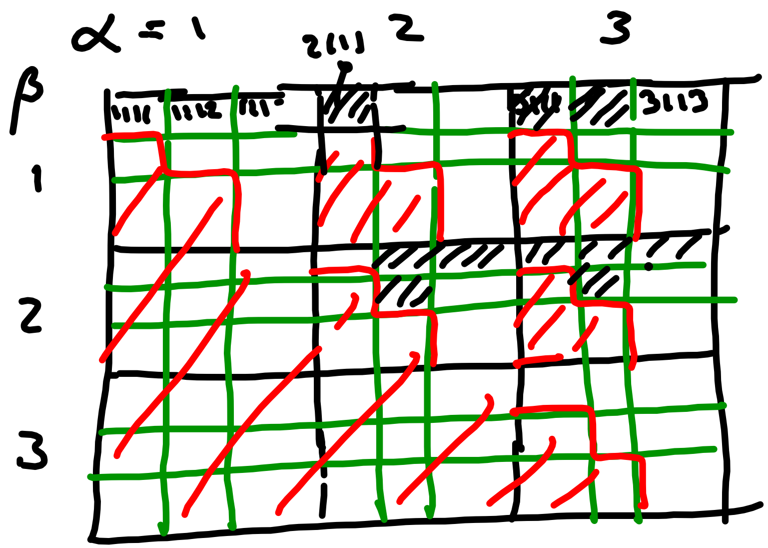
then:

$$\underbrace{C_{\alpha\beta\overbrace{\gamma\delta}^{\text{6 indep.}}}}_{\text{6 independent comb.}} = C_{\beta\alpha\gamma\delta} = C_{\alpha\beta\delta\gamma}$$

Then only 36 components are still independent.

Another property of C is that

$$C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}$$



$6+5+4+3+2+1=$

$=21$

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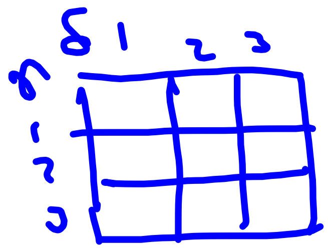

In a 6x6 symmetric matrix the number of independent elements is:

$$21 = \frac{N(N+1)}{2} = \frac{6 \times 7}{2}$$

Because of this physicists arrange the independent components of $C_{\alpha\beta\gamma\delta}$ in a 6x6 symmetric matrix:

11 — 1	23, 32 — 4	$C_{1111} \equiv C_{11}$
22 — 2	13, 31 — 5	$C_{1133} \equiv C_{13}$
33 — 3	21, 12 — 6	$C_{3113} = C_{55}$

Quotient rule

If $A^i{}_j$ and B_{kl} are two tensors we know that

$$K^i{}_{jkl} = A^i{}_j B_{kl} \quad \text{is a tensor}$$

But if we have

$$K_i{}^j A_j = B_i$$

and we know that A_j and B_i are tensors. How do we know if $K_i{}^j$ is a tensor?

Notice that here it seems that K could be the quotient of B and A.

Quotient rule: if we know that B is a tensor and it results from applying an entity K to a generic tensor A we will know that K is also a tensor if the relationship between A, B and K is the same in all other reference frames.

D/ In S': $C'^{ij} A'_j = B'^i$ (A_j and B^i are tensors)

We want to see that C^{ij} is also a tensor.

$$C'^{ij} A'_j = B'^i = \frac{\partial x'^i}{\partial x^j} B^j = \frac{\partial x'^i}{\partial x^j} [\underbrace{C^{jk} A_k}_{B^j}] =$$

$$= \frac{\partial x'^i}{\partial x^j} C^{jk} \frac{\partial x'^l}{\partial x^k} A'_l =$$

$$= \frac{\partial x'^i}{\partial x^l} C^{lk} \frac{\partial x'^j}{\partial x^k} A'_j =$$

$$= \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^j}{\partial x^k} C^{lk} A'_j$$

$$\underbrace{\left(C'^{ij} - \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^j}{\partial x^k} C^{lk}\right)}_{0} A'_j = 0 \Rightarrow C'^{ij} = \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^j}{\partial x^k} C^{lk}$$

tensor of rank 2.

Examples of Tensors:

In 2D, rank 1:

$$V^k = \partial_j T^{jk} = \partial_1 T^{1k} + \partial_2 T^{2k} =$$

$$= \left(\frac{\partial T^{11}}{\partial x^1} + \frac{\partial T^{21}}{\partial x^2}, \frac{\partial T^{12}}{\partial x^1} + \frac{\partial T^{22}}{\partial x^2} \right)$$

rank 2:

$$U^{ijk} T_{jkl} = C^i{}_l = \begin{pmatrix} C^1{}_1 & C^1{}_2 \\ C^2{}_1 & C^2{}_2 \end{pmatrix} = \begin{pmatrix} U^{1jk} T_{jk1} & U^{1jk} T_{jk2} \\ U^{2jk} T_{jk1} & U^{2jk} T_{jk2} \end{pmatrix}$$

$$C^1{}_1 = U^{111} T_{111} + U^{121} T_{211} + U^{112} T_{121} + U^{122} T_{221}$$

Fundamental tensor: metric tensor

g_{ij} is a tensor that will allow us to lower indices (we already saw that with the oblique system). Its inverse g^{ij} will allow us to raise indices. g_{ij} can be any arbitrary symmetric tensor of rank 2 that has an inverse. But we are going to use the so-called "metric tensor" as g_{ij}.

We saw that $g_{ij} = \dfrac{\partial x_i}{\partial x_j}$ for the oblique system.

Consider system S with $\{x^i\}$ and S' with $\{x'^j\}$.

In S we define $d\bar{r} = dr^k$

In S' we have $d\bar{r}' = dr'^k$

contravariant vectors

As vectors we know that $d\bar{r} \equiv d\bar{r}'$ (only the components change)

We will define the basis vectors $\bar{\varepsilon}_i$ in S and $\bar{\varepsilon}'_j$ in S'.

In S:

$$d\bar{r} = dx^{j}\,\bar{\varepsilon}_{j} \quad (1)$$

In cartesian systems

$$\bar{\varepsilon}_{j} \equiv \hat{e}_{j}$$

and

$$d\bar{r}' = dx'^{j}\,\bar{\varepsilon}'_{j} \quad (2)$$

From (1) we see that $\bar{\varepsilon}_{j} = \dfrac{d\bar{r}}{dx^{j}}$

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$$(\varepsilon_{i})_{k} = \frac{dr^{k}}{\partial x^{i}}$$

$$= \frac{\partial x^{k}}{\partial x^{i}} \propto \delta^{k}{}_{i}$$

The component k of ε_i is the derivative of the component k of r with respect to x^i

From (2)

$$\bar{\varepsilon}'_{j} = \frac{d\bar{r}'}{dx'^{j}}$$

Now let's define a rank 2 tensor through the outer product of the basis vectors:

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \equiv (\bar{\varepsilon}_i)_k (\bar{\varepsilon}_j)^k$$

In cartesian coordinates

$$\bar{r} = x^k \hat{e}_k$$

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \frac{\partial \bar{r}}{\partial x^i} \cdot \frac{\partial \bar{r}}{\partial x^j} = \hat{e}_i \cdot \hat{e}_j = \begin{pmatrix} \hat{e}_1 \cdot \hat{e}_1 & \hat{e}_1 \cdot \hat{e}_2 & \hat{e}_1 \cdot \hat{e}_3 \\ \hat{e}_2 \cdot \hat{e}_1 & \hat{e}_2 \cdot \hat{e}_2 & \hat{e}_2 \cdot \hat{e}_3 \\ \hat{e}_3 \cdot \hat{e}_1 & \hat{e}_3 \cdot \hat{e}_2 & \hat{e}_3 \cdot \hat{e}_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \text{ in cartesians.}$$

In this case $g_{ij} = g^{ij} = \mathbb{I}$ which shows why $x^i = x_i$ in cartesian since $x_i = g_{ij} x^j$

Let's see that in general g_{ij} is a tensor:

$$\underset{\substack{\parallel \\ \text{scalar} \\ \text{tensor}}}{ds^2} = \underbrace{d\bar{r} \cdot d\bar{r}}_{dr_k\, dr^k} = \frac{d\bar{r}}{\partial x^i} \partial x^i \frac{\partial \bar{r}}{\partial x^j} \partial x^j = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \partial x^i dx^j$$

$$= g_{ij} \underbrace{dx^i dx^j}_{\text{tensors}}$$

due to the quotient rule g_{ij} is a tensor.

Let's define:

$$g^{ij} = (g_{ij})^{-1}$$

$$\therefore \quad g^{ij} g_{jk} = \delta^{i}{}_{k}$$

Define:

$$g^{ik} A_k = A^i \qquad \text{then}$$

$$g_{ik} A^k = g_{ik} \left[g^{kl} A_l \right] = \delta_i{}^l A_l = A_i$$

Then g_{ij} lowers indices and
g^{ij} raises them.