Metric Tensor

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We saw that

$$g_{ij} = \frac{\partial x_i}{\partial x_j} \quad \text{or} \quad g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j \equiv (\bar{\varepsilon}_i)_k (\bar{\varepsilon}_j)^k$$

Notice that in cartesian coordinates

$\hat{e}_i$ are vectors whose covariant and contravariant components are the same.

Also $g_{ij} = \frac{\partial x_i}{\partial x_j} = \frac{\partial x^i}{\partial x_j} = \delta^i_j$ $\therefore$ in cartesian

$g_{ij} = \mathbb{I}$

We saw that g_{ij} is a tensor of rank 2 using the quotient rule. Now let's see specifically how it transforms from S to S':

In S: $g^{ij} g_{jk} = \delta^i_k$ (1) because $g^{ij} = (g_{ij})^{-1}$.

In S':

$$g'^{ij} g'_{jk} = \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} g^{\alpha\beta} \frac{\partial x^r}{\partial x'^j} \frac{\partial x^\delta}{\partial x'^k} g_{r\delta} =$$

$$= \frac{\partial x'^i}{\partial x^\alpha} \underbrace{\frac{\partial x^r}{\partial x'^j} \frac{\partial x'^j}{\partial x^\beta}}_{\frac{\partial x^r}{\partial x^\beta} = \delta^r_\beta} \frac{\partial x^\delta}{\partial x'^k} g^{\alpha\beta} g_{r\delta} =$$

$$= \frac{\partial x'^{i}}{\partial x^{\alpha}} \, \delta^{\gamma}{}_{\beta} \, \frac{\partial x^{\delta}}{\partial x'^{k}} \, g^{\alpha\beta} g_{\gamma\delta} =$$

$$= \frac{\partial x'^{i}}{\partial x^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{k}} \underbrace{g^{\alpha\beta} g_{\beta\delta}}_{\delta^{\alpha}{}_{\delta}} =$$

$$= \frac{\partial x'^{i}}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x'^{k}} = \frac{\partial x'^{i}}{\partial x'^{k}} = \delta'^{i}{}_{k}$$

Then in S'

$$g'^{ij} g'_{jk} = \delta'^{i}{}_{k}$$

demonstrating that g^{ij} and g_{jk} are tensors of rank 2 and that (1) holds in all systems.

Properties of g_{ij}:

$g_{ij} = g_{ji}$ is symmetric.

since

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \bar{\varepsilon}_j \cdot \bar{\varepsilon}_i = g_{ji}$$

Then

$$g_{ij} A^j = g_{ji} A^j = A_i$$

Examples:

g_{ij} in polar coordinates.

$S: \quad (x^1, x^2) = (\rho, \varphi)$

$S': \quad (x'^1, x'^2) = (x, y)$

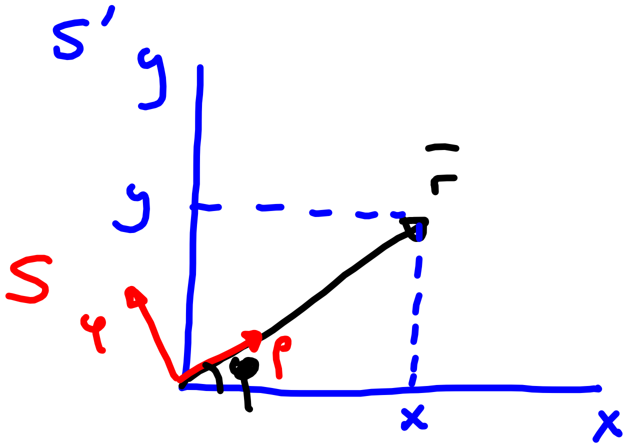
We need to have

$x'^i = x'^i(x^j)$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

$x^j = x^j(x'^i)$ inverse transf.

$$\begin{cases} \rho = (x^2+y^2)^{1/2} \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$



Notice that

$$M^{i}{}_{j} = \frac{\partial x'^{i}}{\partial x^{j}} \quad \text{and} \quad A^{i}{}_{j} = \frac{\partial x^{i}}{\partial x'^{j}}$$

To obtain $g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j$ in S you need to find $\bar{\varepsilon}_\rho$ and $\bar{\varepsilon}_\varphi$.

Notice that

$$\bar{r}' = (x, y) = (\rho\cos\varphi, \rho\sin\varphi) = \rho\cos\varphi\,\hat{e}_1 + \rho\sin\varphi\,\hat{e}_2 \equiv$$

$$\rho\,\hat{\varepsilon}_\rho + \varphi\,\hat{\varepsilon}_\varphi = \bar{r}$$

$$\bar{\varepsilon}_\rho = \frac{d\bar{r}}{d\rho} = \frac{d\bar{r}'}{d\rho} = (\cos\varphi, \sin\varphi)$$ written in cartesian components.

$$\bar{\varepsilon}_\varphi = \frac{\partial \bar{r}}{\partial \varphi} = \frac{\partial \bar{r}'}{\partial \varphi} = (-\rho \sin\varphi, \rho\cos\varphi)$$ in cartesian basis.

Notice that

$|\bar{\varepsilon}_\rho| = 1$ but $|\bar{\varepsilon}_\varphi| = \rho \neq 1$!

Now

$$g_{ij} = \bar{\varepsilon}_i \cdot \bar{\varepsilon}_j = \begin{pmatrix} \bar{\varepsilon}_\rho \cdot \bar{\varepsilon}_\rho & \bar{\varepsilon}_\varphi \cdot \bar{\varepsilon}_\varphi \\ \bar{\varepsilon}_\varphi \cdot \bar{\varepsilon}_\rho & \bar{\varepsilon}_\varphi \cdot \bar{\varepsilon}_\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix} \neq \mathbb{I}$$

g_{ij} is diagonal because $\bar{\rho} \perp \bar{\varphi}$ but it is not $\mathbb{I}$ because the basis is non-cartesian.

Then now

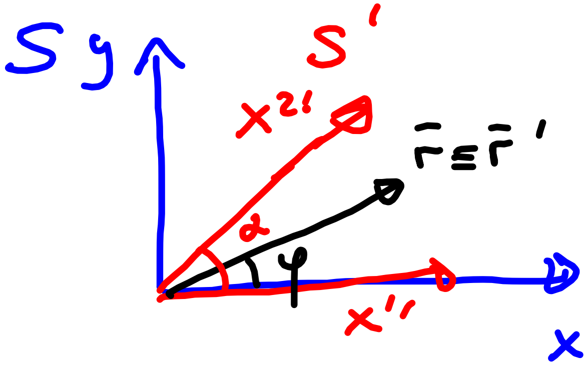
$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix} \quad \text{so that} \quad g^{ij} g_{jk} = \delta^{i}{}_{k}$$

Now consider the oblique system.

We saw that $g'_{ij} = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}$ using that

$$g'_{ij} = \frac{\partial x'_i}{\partial x'^j}$$

Now let's find $g'_{ij} = \bar{\varepsilon}'_i \cdot \bar{\varepsilon}'_j$



$$\begin{cases} x_1 = x'^1 + x'^2 \cos\alpha \\ x_2 = x'^2 \sin\alpha \end{cases}$$

$\bar{\varepsilon}'_1 = \hat{e}'_1 = (1,0)$ and $\bar{\varepsilon}'_2 = \hat{e}'_2 = (\cos\alpha, \sin\alpha)$

In S: $g_{ij} = g^{ij} = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In S': $g'_{ij} = \bar{\varepsilon}'_i \cdot \bar{\varepsilon}'_j = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}$

Same result as with

$$\frac{\partial x'_i}{\partial x'^j} = g_{ij}$$

$$g'^{ij} = (g'_{ij})^{-1} = \frac{1}{\operatorname{sen}^2\alpha}\begin{pmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{pmatrix}$$

Notice that

$$x'_i = g_{ij}\, x'^j$$

Levi-Civita Tensor

Not in first midterm

In 3D:

$$\underbrace{\varepsilon^{ijk} = \varepsilon_{ijk}}_{\text{in cartesians}} = \begin{cases} 1 & \text{if } i \neq j \neq k \text{ cyclic order} \\ -1 & \text{if } i \neq j \neq k \text{ not in cyclic order} \\ 0 & \text{otherwise.} \end{cases}$$

anti-symmetric in all pairs of components

$\varepsilon_{123} = \varepsilon_{231} = 1$

$\varepsilon_{213} = \varepsilon_{321} = -1$

$\varepsilon_{112} = \varepsilon_{233} = 0$

- Rank 3 tensor
- Antisymmetric in all pair of indices.
- $3^3 = 27$ components.
- It has 6 non-zero components.
- Only one independent component (knowing that $\varepsilon_{123} = 1$ you can get all the others).

It can be generalized to N dimensions, with exactly the same properties.

Properties of ε^{ijk}:

In 3D:

$$\varepsilon_{pqr} = \hat{x}^p \cdot (\hat{x}^q \times \hat{x}^r)$$

$\hat{x}^n$: coordinate unit vector. (orthogonal)

$$\det M = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = m_{1i}\, m_{2j}\, m_{3k}\, \varepsilon_{ijk} =$$

$$m_{11}\, m_{22}\, m_{33}\, \varepsilon_{123} + \cdots$$

expand using only the 6 $\varepsilon_{ijk} \neq 0$ and you'll get the result.

In N dimensions:

$$\det M = \begin{vmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{vmatrix} = m_{1i}\, m_{2j} \cdots m_{nz}\, \varepsilon_{ij\ldots z}$$

Also in 3D:

$$\det M \; \varepsilon_{\alpha\beta\gamma} = m_{\alpha i}\, m_{\beta j}\, m_{\gamma k}\, \varepsilon_{ijk}$$

$$\parallel$$

$$m_{1i}\, m_{2j}\, m_{3k}\, \varepsilon_{ijk}\, \varepsilon_{\alpha\beta\gamma}$$

(In homework use the possible values of $\alpha\beta\gamma$ to show that it is true).

Cross-product (in 3D)

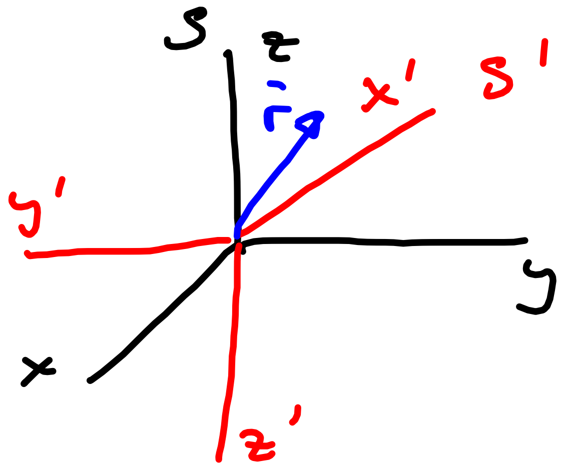
$$\bar{C} = \bar{A} \times \bar{B} \longrightarrow C_i = \varepsilon_{ijk} A^j B^k$$

Further classification of vectors
(and tensors).

Polar and axial vectors:

In physics it is important to study how quantities transform under reflections and inversions.

Inversion!



$$x'^i = -x^i$$

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$|\det M^i{}_j| = -1$$

Let's see how r^i transforms:

$$\bar{r} = (x, y, z) = (-x', -y', -z') = \bar{r}'$$

I'm using $\hat{e}_i$ basis!

A vector is "odd" under an inversion because its coordinates become negative.

Any vector that transforms like r^i under an inversion is called polar.

Now consider:

$$\bar{C} = \bar{A} \times \bar{B}$$

$\bar{A}$ and $\bar{B}$ are polar.

In components:

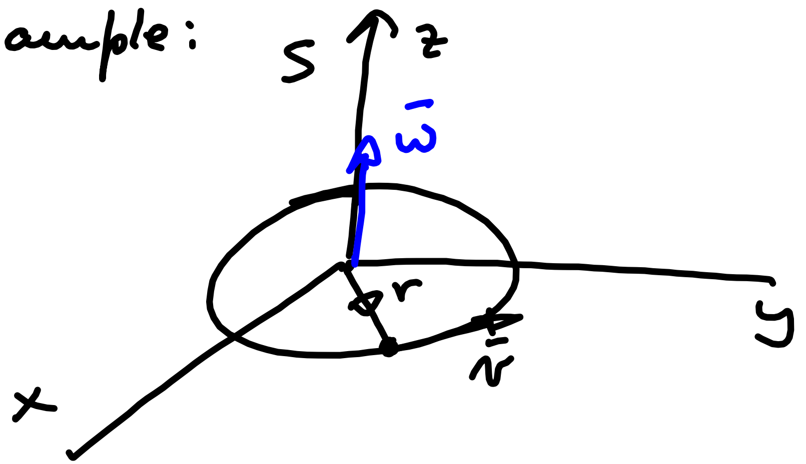
$$C_i = A^j B^k - A^k B^j$$

In S' (inverted system).

$$A^{j\prime} = -A^j \quad \text{and} \quad B^{k\prime} = -B^k$$

Then we see that $C_i' = C_i$ no sign change.

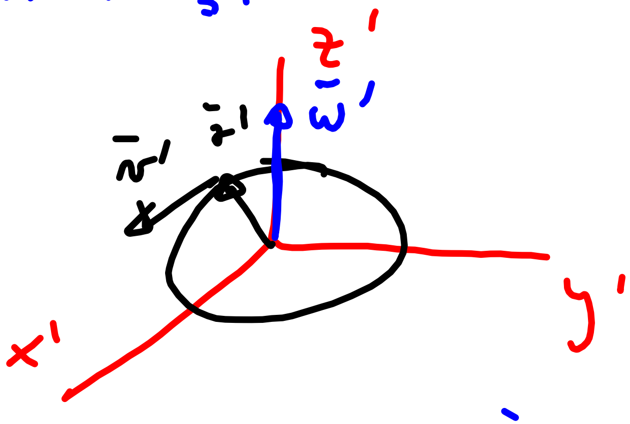
Example:



$$\bar{v} = \bar{\omega} \times \bar{r}$$

$$\bar{\omega} = \frac{\bar{r} \times \bar{v}}{|\bar{r}|^2}$$

In the inverted system S' I will see:



$\bar{\omega}$ is an axial vector.