Polar (vector) and Axial (pseudovector) tensors. 9/30.

$$\bar{L} = \bar{r} \times \bar{p}$$

$\bar{L}$ is a pseudovector or axial vector because both $\bar{r}$ and $\bar{p}$ are polar vectors.

Notice that in tensor notation

$$L_i = \varepsilon_{ijk} r^j p^k$$

We know that a pseudovector transforms like:

$$C'^{i} = \det M^{i}{}_{j} \frac{dx'}{dx^j} C^{j}$$

How does ε^{ijk} transform?

tensor of rank 3

$$\varepsilon'^{ijk} \overset{?}{=} \frac{\partial x'^{i}}{\partial x^{\alpha}} \frac{\partial x'^{j}}{\partial x^{\beta}} \frac{\partial x'^{k}}{\partial x^{\gamma}} \varepsilon^{\alpha\beta\gamma} =$$

$$= M^{i}{}_{\alpha} M^{j}{}_{\beta} M^{k}{}_{\gamma} \varepsilon^{\alpha\beta\gamma} =$$

$$= \underbrace{\det M}\, \varepsilon^{ijk}$$

this indicates that it transforms like a pseudo tensor which is isotropic.

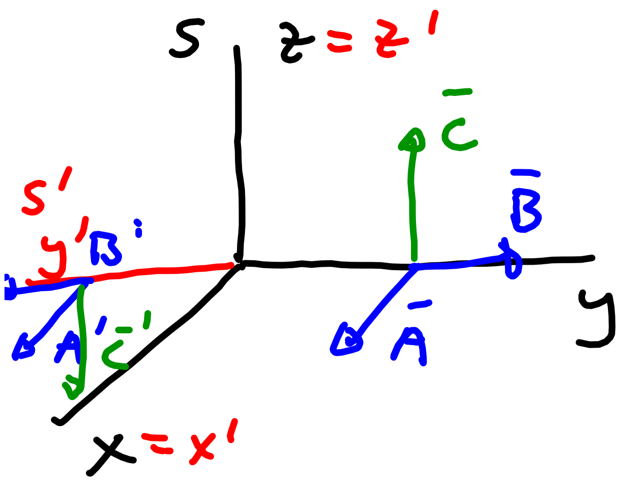
ε^{ijk} is an isotropic tensor because it has the same form in every system.

Then ε^{ijk} is a pseudotensor and if it appears in any expression together with vectors as in

$$C^i = \varepsilon^{ijk} A_j B_k$$ with A_j, B_k polar vectors.

then C^i, due to ε^{ijk}, will be a pseudovector.

Reflections: this is also an improper transformation with $\det M = -1$.



In $S: (x, y, z)$

$S': (x' = x, y' = -y, z' = z)$

$$M^i{}_j = \frac{\partial x'^i}{\partial x^j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det M = -1$$

$$\bar{A} \times \bar{B} = \bar{C}$$

Vectors $\bar{A}$ and $\bar{B}$ look like reflected on a mirror, but pseudovector $\bar{C}$ appears inverted.

We see again that

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

A^j is a vector

but

$$C'^i = \det M \, \frac{\partial x'^i}{\partial x^j} C^j$$

C^j is a pseudo vector.

Let's demonstrate that C^i transforms like a pseudovector if $C^i = \varepsilon^{ijk} A_j B_k$

$$C'^i = \varepsilon'^{ijk} A'_j B'_k = \det M \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta} \frac{\partial x'^k}{\partial x^\gamma}$$

$$\frac{\partial x^\delta}{\partial x'^j} \frac{\partial x^\varepsilon}{\partial x'^k} \varepsilon^{\alpha\beta\gamma} A_\delta B_\varepsilon =$$

$$\frac{\partial x^\delta}{\partial x'^j} \frac{\partial x'^j}{\partial x^\beta} = \delta^\delta{}_\beta$$

$$\frac{\partial x^\varepsilon}{\partial x'^k} \frac{\partial x'^k}{\partial x^\gamma} = \delta^\varepsilon{}_\gamma$$

$$= \det M \, \delta^\delta{}_\beta \delta^\varepsilon{}_\gamma \frac{\partial x'^i}{\partial x^\alpha} \varepsilon^{\alpha\beta\gamma} A_\delta B_\varepsilon =$$

$$= \det M \frac{\partial x'^i}{\partial x^\alpha} \underbrace{\varepsilon^{\alpha\beta\gamma} A_\beta B_\gamma}_{C^\alpha} = \det M \frac{\partial x'^i}{\partial x^\alpha} C^\alpha$$

pseudotensor.

We can generalize this to tensors of any rank:

Tensors

scalar

$$S' = S$$

vector

$$C'^i = \frac{\partial x'^i}{\partial x^j} C^j$$

rank 2 tensor:

$$A'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} A^{kl}$$

Pseudotensors.

pseudoscalar:

$$S' = \det M \quad S$$

pseudovector:

$$C'^i = \det M \; \frac{\partial x'^i}{\partial x^j} C^j$$

rank 2 pseudotensor:

$$A'^{ij} = \det M \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} A^{kl}$$

If you do the outer product of tensors and pseudotensors you will get that:

$$T \otimes T = T$$

$$T \otimes P = P$$

$$P \otimes P = T$$

because

$$P' \otimes P' = \underbrace{(\det M)^2}_{1} \underbrace{P \otimes P}_{\text{tensor}}$$

Dual tensors:

Are tensors of lower rank that can be obtained from the independent components of anti-symmetric tensors.

For example if C^{jk} is an anti-symmetric tensor of rank 2 in 3D we can construct a tensor of rank 1 with its independent components so that

$$C_i = \frac{1}{2} \varepsilon_{ijk} C^{jk}$$

you see that if C^{jk} is a tensor then C_i is a pseudo tensor.

$$C^{jk} = \begin{pmatrix} 0 & C^{12} & C^{13} \\ -C^{12} & 0 & C^{23} \\ -C^{13} & -C^{23} & 0 \end{pmatrix}$$

$$C_1 = \frac{1}{2} \varepsilon_{123} C^{23} + \frac{1}{2} \varepsilon_{132} C^{32} = \frac{1}{2} C^{23} + \frac{1}{2}(-1)(-C^{23})$$

$$= C^{23}$$

$$C_2 = \frac{1}{2} \varepsilon_{231} C^{31} + \frac{1}{2} \varepsilon_{213} C^{13} = -C^{13}$$

$$C_3 = \frac{1}{2} \varepsilon_{312} C^{12} + \frac{1}{2} \varepsilon_{321} C^{21} = C^{12}$$

$$C_i = (C^{23}, -C^{13}, C^{12})$$

Notice that all the pseudovectors that in physics arise from cross products can be thought as the dual representation of antisymmetric tensors of rank 2:

Ex:
$$\bar{L} = \bar{r} \times \bar{p}$$

$$L_i = \varepsilon_{ijk} r^j p^k = \frac{1}{2} \varepsilon_{ijk} (r^j p^k - r^k p^j)$$

Now
$$L^{jk} = \begin{pmatrix} 0 & r^1p^2 - r^2p^1 & r^1p^3 - r^3p^1 \\ & 0 & r^2p^3 - r^3p^2 \\ AS & & 0 \end{pmatrix}$$

Notice that the above, i.e. associating a rank 2 tensor with a vector via dual representation only happens in 3D.

In 2D consider the antisymmetric rank 2 tensor A^{ij}:

$$A^{ij} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$$

Notice that we have only one independent component.

$A = \frac{1}{2} \varepsilon_{ij} A^{ij}$ one component. We need two to make a vector in 2D : $V^i = (V^1, V^2)$.

In $N=4$

T^{ij} anti-symmetric has 6 independent components more than the 4 we need to make a vector.

$C^{ij} = \frac{1}{2}\varepsilon_{ijkl}\, T^{kl}$ the dual in 4D is a tensor of rank 2.

Other examples of dual tensors:

In 3D consider:

$$V^{ijk} = A^i B^j C^k$$

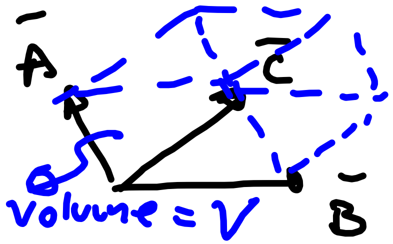
A^i, B^j, C^k are vectors.

V^{ijk}: rank 3 tensor.

Define a pseudoscalar as:

$$V = \varepsilon_{ijk} V^{ijk} = \varepsilon_{ijk} A^i B^j C^k = A^i \underbrace{\varepsilon_{ijk} B^j C^k}_{(\bar{B} \times \bar{C})_i}$$

$$= \bar{A} \cdot (\bar{B} \times \bar{C})$$



We see that the volume in an inverted or reflected system will change sign so it is a pseudoscalar.

Use of vector notation to prove vector formulas:

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = ?$$

Very hard to calculate by components.

In tensor notation:

$$\varepsilon^{lmi}\partial_m \varepsilon_{ijk}\partial^j A^k = \underbrace{\varepsilon^{lmi}\varepsilon_{ijk}}_{\underbrace{\varepsilon^{ilm}\varepsilon_{ijk}}_{\delta^l_j\delta^m_k - \delta^l_k\delta^m_j}} \partial_m \partial^j A^k =$$

$$= \delta^l_j \delta^m_k \partial_m \partial^j A^k - \delta^l_k \delta^m_j \partial_m \partial^j A^k =$$

$$= \partial_k \partial^l A^k - \partial_j \partial^j A^l = \partial^l \partial_k A^k - \partial_j \partial^j A^l = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2 \bar{A}$$

Another identity that you can obtain:

$\bar{\nabla}\cdot(\bar{a}\times\bar{b}) = ?$ $\bar{a}, \bar{b}$: vectors.

$$\partial_i \, \varepsilon^{ijk} a_j b_k = \varepsilon^{ijk}(\partial_i a_j)\, b_k +$$

$$+ \varepsilon^{ijk} a_j \partial_i b_k = b_k \underbrace{\varepsilon^{kij} \partial_i a_j}_{(\bar{\nabla}\times\bar{a})^k} +$$

$$+ a_j \underbrace{(-\varepsilon^{jik}) \partial_i b_k}_{(-\bar{\nabla}\times\bar{b})^j} =$$

$\varepsilon^{ijk} = \varepsilon^{jki} = -\varepsilon^{jik}$

$$= \bar{b}\cdot(\bar{\nabla}\times\bar{a}) - \bar{a}\cdot(\bar{\nabla}\times\bar{b})$$