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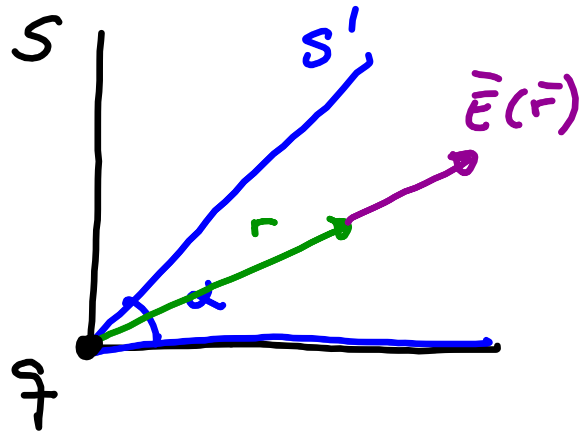
Components of the electric field of a point charge in the oblique system.



In S:

$$V(\bar{r}) = \frac{q}{4\pi\varepsilon_0 r} = \frac{q}{4\pi\varepsilon_0 (x_i x^i)^{1/2}}$$

$$\bar{E} = -\bar{\nabla} V$$

In S': $V'(\bar{r}') = \dfrac{q}{4\pi\varepsilon_0 r'} = \dfrac{q}{4\pi\varepsilon_0 (x'_i x'^i)^{1/2}}$

Last time we found that

$$x'_i = g_{ij} x'^j \quad (1) \qquad g_{ij} = \begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}$$

In S':

$$E'_j = -\partial'_j V' = -\frac{\partial}{\partial x'^j} V' = -\frac{q}{4\pi\varepsilon_0} \frac{\partial}{\partial x'^j} \left[\frac{1}{(x'_i x^{i\prime})^{1/2}} \right] =$$

$$= +\frac{q}{4\pi\varepsilon_0 2 \underbrace{(\underbrace{x'_i x^{i\prime}}_{r'^2 = r^2})^{3/2}}_{r^3}} \left[\underbrace{\frac{\partial x'_i}{\partial x'^j}}_{\frac{\partial [g_{ik} x'^k]}{\partial x'^j}} x^{i\prime} + x'_i \underbrace{\frac{\partial x^{i\prime}}{\partial x'^j}}_{\delta^i{}_j} \right] =$$

using (1)

Notice that

$$\frac{\partial g_{ik} x'^k}{\partial x'^j} = g_{ik} \frac{\partial x'^k}{\partial x'^j} = g_{ik} \delta^k{}_j$$

Then:

$$E'_j = \frac{q}{4\pi\varepsilon_0 r^3 2}\left[\delta^k{}_j \underbrace{g_{ik} x'^i}_{x'_k \text{ (from 1)}} + x'_i \delta^i{}_j\right] =$$

$$= \frac{q}{4\pi\varepsilon_0 r^3 2}\left[\underbrace{x'_j + x'_j}_{2x'_j}\right] = \frac{q}{4\pi\varepsilon_0 r^3} x'_j$$

covariant components.

perp. proj. of $\bar{E}$.

Several physical properties result from the contraction (scalar product) of covariant with contravariant vectors.

Ex: Work

$$W = \bar{F} \cdot d\bar{s} = q\, E_j\, ds^j$$

as matrices means:

$$(E_1, E_2, E_3)\begin{pmatrix} ds^1 \\ ds^2 \\ ds^3 \end{pmatrix}$$

Several operators and their tensor notation:

$$\frac{\partial}{\partial x^i} \equiv \partial_i$$

covariant components of the derivative.

"prototype covariant vector"

Gradient:

$$\bar{\nabla} F = \bar{B} \equiv B_i = \frac{\partial F}{\partial x^i} \equiv \partial_i F$$

scalar field

Total differential:

$$dF = \bar{\nabla} F \cdot d\bar{r} = \partial_i F \, dr^i \qquad \text{scalar}$$

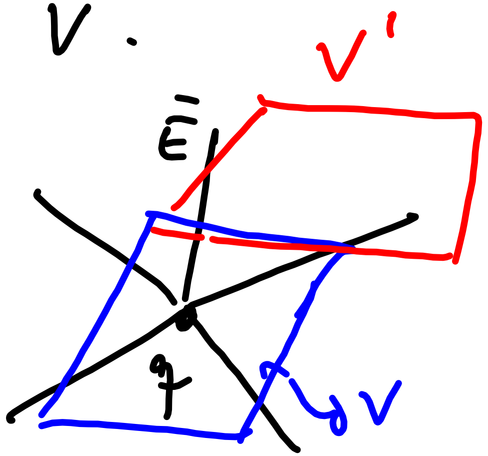
Divergence:

$$\bar{\nabla} \cdot \bar{G} = \partial_i G^i \qquad \text{scalar} \quad (G^i : \text{vector})$$

$$\| $$

$$\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

A field $\bar{G}$ has non-zero divergence in a volume V if it has a singularity inside V.



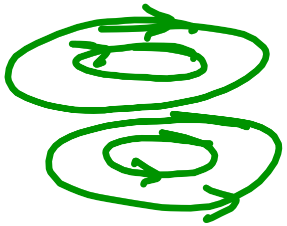
$\bar{\nabla}.\bar{E} \propto q$ inside V.

$\therefore$ in V $\bar{\nabla}.\bar{E} \neq 0$.

In V' $\bar{\nabla}.\bar{E} = 0$. Because there is no charge inside V'!

If there are no singularities in V then $\bar{\nabla}.\bar{G} = 0$.

A field that has zero divergence is called solenoidal.
Ex: Magnetic field (no monopoles).



solenoidal field.

— x —

Tensor notation to obtain useful expressions in 3D.

1) $\bar{\nabla}\cdot\bar{r} = \partial_i x^i = 3$ in N dimensions $\partial_i \cdot x^i = N$.

$$\bar{\nabla}\cdot\bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

2) $\bar{\nabla} \cdot [\bar{r} f(r)] = ?$ $f(r)$: central force (scalar)

$$\partial_i r^i f(r) = \partial_i r^i f\left((x_i x^i)^{1/2}\right) =$$

$$= \underbrace{\partial_i r^i}_{3} f(r) + r^i \partial_i f\left[(x_i x^i)^{1/2}\right] =$$

$$= 3 f(r) + r^i \frac{\partial f}{\partial r} \frac{\partial r}{\partial x^i} =$$

$x_j = g_{jk} x^k$ always.

$$= 3 f(r) + r^i \frac{\partial f}{\partial r} \frac{\partial\left[(x_j x^j)^{1/2}\right]}{\partial x^i} =$$

$$= 3 f(r) + r^i \frac{\partial f}{\partial r} \frac{1}{2} \frac{1}{(x_j x^j)^{1/2}} \left(\frac{\partial x_j}{\partial x^i} x^j + x_j \underbrace{\frac{\partial x^j}{\partial x^i}}_{\delta^j{}_i} \right)$$

$$= 3f(r) + x^i \frac{1}{2} \frac{\partial f}{\partial r} \frac{1}{r} \left[g_{jk} \overbrace{\frac{\partial x^k}{\partial x^i}}^{\delta^k_i} x^j + \right.$$

$$\left. + x_j \delta_i^j \right] =$$

$$g_{jk} x^j = x_k$$

$$= 3f(r) + x^i \frac{\partial f}{\partial r} \frac{1}{2r} \left[x_k \delta^k_i + x_i \right] =$$

$$= 3f(r) + x^i \frac{\partial f}{\partial r} \frac{1}{2r} \left[x_i + x_i \right] =$$

$$= 3f(r) + \frac{\partial f}{\partial r} \frac{1}{2r} 2 \underbrace{x^i x_i}_{r^2} = \boxed{3f(r) + r \frac{\partial f}{\partial r}}$$

Application of (2)! $\quad r^{n-1} = f(r)$

$$\bar{\nabla} \cdot (\bar{r}\, r^{m-1}) = 3\, r^{n-1} + r \frac{\partial r^{n-1}}{\partial r} =$$

$$= 3 r^{m-1} + r\,(m-1)\, r^{m-2} = 3 r^{m-1} + r^{m-1}(m-1) =$$

$$= (2+m)\, r^{m-1}$$

Curl

$$\bar{\nabla} \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{i} +$$

$$+ \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{k}$$

Example:

$\bar{B} = \bar{\nabla} \times \bar{A}$ ($\bar{A}$: vector potential)

$\bar{\nabla} \times \bar{E} = 0$ (for electrostatic field).

If $\bar{\nabla} \times \bar{G} = 0$ then $\bar{G}$ is irrotational.

Example:

$$\bar{\nabla} \times (\bar{r} f(\bar{r})) = ?$$

central force

Using the vector identity (that we will demonstrate later):

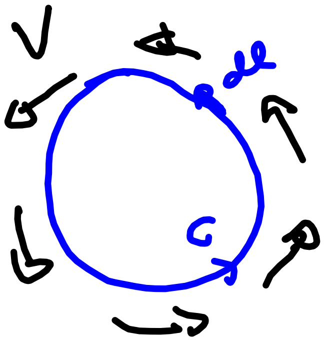
scalar field

$$\bar{\nabla} \times (f \bar{V}) = f \bar{\nabla} \times \bar{V} + (\bar{\nabla} f) \times \bar{V}$$

vector field

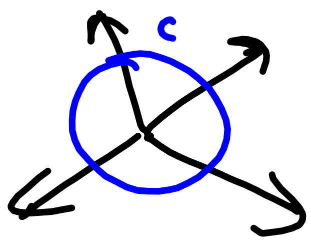
$$\bar{\nabla} \times (\bar{r}\, f(r)) = f\, \underbrace{\bar{\nabla} \times \bar{r}}_{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0} + \underbrace{\underbrace{(\bar{\nabla} f)}_{\parallel \bar{r}} \times \bar{r}}_{0} = 0$$

Physical interpretation of the curl:



$$\oint_C \bar{V} \cdot \bar{dl} \neq 0$$

V is irrotational.



$$\oint_c \bar{V} \cdot \bar{dl} = 0$$

Laplacian

$$\underbrace{\nabla^2 \varphi}_{\text{scalar}} = \underbrace{\bar{\nabla} \cdot (\underbrace{\bar{\nabla} \varphi}_{\substack{\text{gradient} \\ \text{(vector)}}})}_{\text{divergence}} = \underbrace{\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}}_{\text{scalar}}$$

In tensor notation:

$$\nabla^2 = \partial_i \partial^i = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x_i}$$

contravariant components

$$\partial^i = \frac{\partial}{\partial x_i}$$

covariant

Contravariant components:

$x_j = g_{ji} x^i$

$\frac{\partial x_j}{\partial x^i} = g_{ji}$

$$\boxed{\partial^i V} = \frac{\partial V}{\partial x_i} = \frac{\partial V}{\partial x^j} \underbrace{\frac{\partial x^j}{\partial x_i}}_{(g_{ij})^{-1} \equiv g^{ji}} =$$

$$= g^{ji} \frac{\partial V}{\partial x^j} = \boxed{g^{ji} \partial_j V}$$

Then $\partial^i = g^{ji} \partial_j$

$$g_{ij} g^{jk} = \frac{\partial x_i}{\partial x^j} \frac{\partial x^j}{\partial x_k} = \frac{\partial x_i}{\partial x_k} = \delta^i{}_k \Rightarrow g^{jk} = (g_{ij})^{-1}$$

Then:

$$\nabla^2 = \partial_i \partial^i = g^{ij} \partial_i \partial_j$$

Examples:

$$\bar{E} = -\bar{\nabla} V \qquad \text{if} \qquad \bar{\nabla} \cdot \bar{E} = 0$$ (no charges)

$$0 = \bar{\nabla} \cdot \bar{E} = -\bar{\nabla}(\bar{\nabla} V) = -\nabla^2 V$$

or $\boxed{\nabla^2 V = 0}$ Laplace's equation