

Midterm Exam

P571

October 2, 2014

SOLUTION:

a) We can obtain $M^i_j = \frac{\partial x'^i}{\partial x^j}$ from the information provided since we know that $x'^1 = 2x^1 - x^2$ and $x'^2 = -x^1 + 2x^2$. Then,

$$M^i_j = \frac{\partial x'^i}{\partial x^j} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (1)$$

b) The inverse transformation matrix is given by $A^i_j = \frac{\partial x^i}{\partial x'^j}$. The inverse transformation can be obtained from the information provided and we find that $x^1 = \frac{2}{3}x'^1 + \frac{1}{3}x'^2$ and $x^2 = \frac{1}{3}x'^1 + \frac{2}{3}x'^2$. Then,

$$A^i_j = \frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \quad (2)$$

By multiplying M by A we see that

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3)$$

as expected.

c) The covariant basis in S is given by:

$$\mathbf{e}_1 = (1, 0), \quad (4)$$

$$\mathbf{e}_2 = (0, 1), \quad (5)$$

d) Since we need to find the vectors of the covariant basis in S' we know that:

$$\mathbf{e}'_i = \frac{\partial x^j}{\partial x'^i} \mathbf{e}_j, \quad (6)$$

which corresponds to the multiplication of the coefficients of \mathbf{e}_j as a row vector by the matrix A . Then, we obtain:

$$\mathbf{e}'_1 = \frac{2}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2, \quad (7)$$

and

$$\mathbf{e}'_2 = \frac{1}{3}\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2. \quad (8)$$

Thus, in cartesian components we have found that $\mathbf{e}'_1 = (\frac{2}{3}, \frac{1}{3})$ and $\mathbf{e}'_2 = (\frac{1}{3}, \frac{2}{3})$.

Another way: We also can use the fact that $\mathbf{r} = \mathbf{r}'$ as vectors and then

$$\mathbf{r} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 = (2x^1 - x^2)\mathbf{e}_1 + (-x^1 + 2x^2)\mathbf{e}_2 = \mathbf{r}' = x'^1\mathbf{e}'_1 + x'^2\mathbf{e}'_2 \quad (9)$$

From Eq.(9) we see that $\mathbf{e}'_i = \frac{\partial \mathbf{r}'}{\partial x'^i} = \frac{\partial \mathbf{r}}{\partial x'^i}$. Then in cartesian components we see that $e'_i = \frac{\partial x^j}{\partial x'^i} e_j$ and we can obtain for e'_i the same values that we found above.

d-i) In order to see if the covariant basis vectors in S' are orthogonal we need to calculate their scalar product which is simple because we have found their cartesian components then:

$$\mathbf{e}'_1 \cdot \mathbf{e}'_2 = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}. \quad (10)$$

Since the result is non zero we see that the basis vectors in S' are not orthogonal.

d-ii) In order to see if the covariant basis vectors in S' are normal we need to calculate their norm by taking the squared root of the scalar product of each vector with itself which is simple because we have found their cartesian components then:

$$|\mathbf{e}'_i| = \sqrt{\mathbf{e}'_i \cdot \mathbf{e}'_i} = \sqrt{\frac{4}{9} + \frac{1}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}, \quad (11)$$

and

$$|\mathbf{e}'_2| = \sqrt{\mathbf{e}'_2 \cdot \mathbf{e}'_2} = \sqrt{\frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}. \quad (12)$$

Since the norm of the basis vectors is not 1, then the basis vectors in S' are not normal.

e)

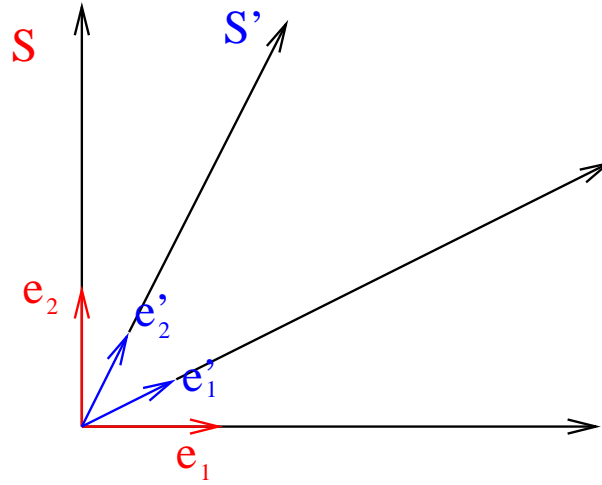


FIG. 1: Covariant basis vectors \mathbf{e}'_i and \mathbf{e}_i .

f) We know that

$$g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = \begin{pmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{pmatrix}. \quad (13)$$

g) We know that

$$\mathbf{e}'_i \cdot \mathbf{e}'^j = \delta_i^j, \quad (14)$$

then we can assume that $\mathbf{e}'^1 = (a, b)$ and $\mathbf{e}'^2 = (c, d)$. Plugging this and the values for \mathbf{e}'_i obtained in (d) in Eq.(14) we obtain four equations with the four unknowns $a, b, c,$ and d . Solving the equations we find:

$$\mathbf{e}'^1 = (2, -1), \quad (15)$$

and

$$\mathbf{e}'^2 = (-1, 2). \quad (16)$$

Here you also could use $g'^{ij} = (g'_{ij})^{-1}$ to calculate the contravariant components with

$$g'^{ij} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}.$$

g-i)

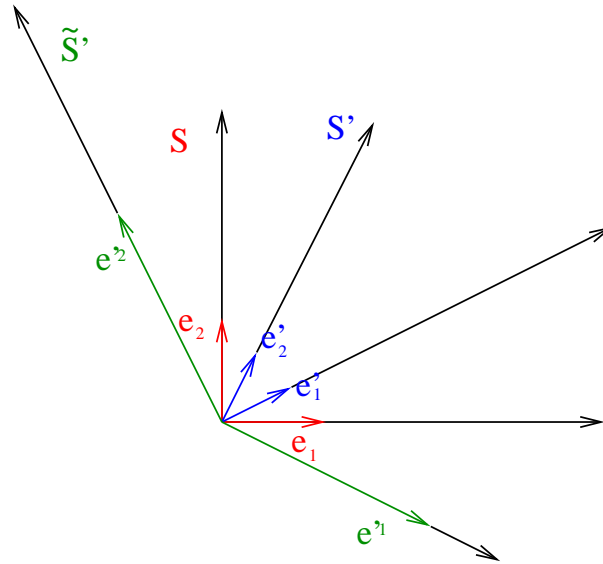


FIG. 2: Contravariant basis vectors \mathbf{e}'^i .

h) Since

$$\mathbf{r}' = x'_1 \mathbf{e}'^1 + x'_2 \mathbf{e}'^2 = x'_1(2, -1) + x'_2(-1, 2) = x^1 \mathbf{e}_1 + x^2 \mathbf{e}_2 = (x^1, x^2) = \mathbf{r}, \quad (17)$$

comparing components we find that $2x'_1 - x'_2 = x^1$ and $-x'_1 + 2x'_2 = x^2$ then

$$x'_1 = \frac{2}{3}x^1 + \frac{1}{3}x^2, \quad (18)$$

and

$$x'_2 = \frac{1}{3}x^1 + \frac{2}{3}x^2. \quad (19)$$

Notice that another way of obtaining the same result is by using the metric tensor found in (f) and using that

$$x'_i = g'_{ij} x'^j. \quad (20)$$

i) The problem provides the contravariant components p'^i and k'^i . To obtain the covariant components we need to use that the covariant components of a tensor are obtained via the metric tensor calculated in part (f). Thus, for a generic vector \mathbf{r} in system S' we know that

$$r'_i = g'_{ij} r'^j. \quad (21)$$

Then we find that

$$p'_i = \left(\frac{11}{9}, \frac{7}{9}\right), \quad (22)$$

and

$$k'_i = \left(-\frac{2}{3}, -\frac{1}{3}\right). \quad (23)$$

j) To calculate the magnitude of the vectors in S' we need to contract their covariant components with the contravariant ones. Then

$$|\mathbf{p}'| = \sqrt{p'_i p'^i} = \frac{\sqrt{26}}{3}, \quad (24)$$

and

$$|\mathbf{k}'| = \sqrt{k'_i k'^i} = 1. \quad (25)$$

The angle α between the two vectors is obtained from the scalar product:

$$\cos \alpha = \frac{k'_i p'^i}{|\mathbf{k}'||\mathbf{p}'|} = -\frac{5}{\sqrt{26}}. \quad (26)$$

Then $\alpha = \cos^{-1}\left(-\frac{5}{\sqrt{26}}\right) = 168.7^\circ$.

j-i) My answer in S would have been the same since both the magnitude of the vectors and the angle between them are tensors of rank 0.

To verify this let's calculate k^i and p^i in S . We know that the transformation for a contravariant vector \mathbf{r} is given by $r^i = \frac{\partial x^i}{\partial x'^j} r'^j$ then we find that

$$k^i = (-1, 0), \quad (27)$$

and

$$p^i = \left(\frac{5}{3}, \frac{1}{3}\right), \quad (28)$$

which allows us to verify that $|\mathbf{p}| = \frac{\sqrt{26}}{3}$, $|\mathbf{k}| = 1$, and $\alpha = 168.7^\circ$.

k) The tensor T'^{ijklm} has rank 4 because it has 4 indices.

k-i) Since it is defined in two dimensional space the tensor has $2^4 = 16$ components.

k-ii) We notice that the tensor is symmetric under the exchange of ij and under the exchange of lm because $k^i k^j = k^j k^i$ and $p^l p^m = p^m p^l$ then instead of 4 independent ij values there are 3 and instead of 4 independent lm values there are 3, which means that there are $3 \times 3 = 9$ independent components.

l) Since we are contracting the first index with the second and the third with the fourth we see that $T_i^i j^j$ is a tensor of rank 0, i.e., it is a scalar.

l-i) Notice that

$$T_i^i j^j = k'_i k'^i p'_j p'^j = |\mathbf{k}'|^2 |\mathbf{p}'|^2 = \frac{26}{9}. \quad (29)$$

Since it is a scalar we find that

$$T_i^i j^j = k_i k^i p_j p^j = |\mathbf{k}|^2 |\mathbf{p}|^2 = \frac{26}{9}. \quad (30)$$

As expected both quantities have the same value.