Midterm Exam II

P571
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SOLUTION:

Problem 1:

a)

$$A_k = \epsilon_{kji} \frac{m^j x^i}{(x_s x^s)^{3/2}}. \tag{1}$$

b) As we see from the RHS of Eq.(1) the only free index,k, is covariant. Thus, we have obtained the covariant components of $\mathbf{A}$.

c) Now we need to consider how $\mathbf{A}$ transforms from a system S to a system S'. It is a tensor if the transformation has none or an even number of (det M) factors, where M is the transformation matrix and it is a pseudotensor when the number of (det M) factors is odd. From Eq.(1) we see that there will be two (det M) factors. One coming from the transformation of the pseudotensor m^j and the other coming from the transformation of the Levi-Civita tensor. Thus, $\mathbf{A}$ is a tensor.

d)

$$(\nabla \times \mathbf{A})^i = \epsilon^{ijk} \partial_j \epsilon_{ktu} \frac{m^t x^u}{(x_s x^s)^{3/2}}. \tag{2}$$

e) Now we need to consider how $\nabla \times \mathbf{A}$ transforms from a system S to a system S'. It is a tensor if the transformation has none or an even number of (det M) factors, where M is the transformation matrix and it is a pseudotensor when the number of (det M) factors is odd. From Eq.(2) we see that there will be three (det M) factors. One coming from the transformation of the pseudotensor m^j and the other two coming from the transformation of the two Levi-Civita tensors. Thus, $\nabla \times \mathbf{A}$ is a pseudotensor.

f)

$$(\nabla \times \mathbf{A})^i = \epsilon^{ijk} \partial_j \epsilon_{ktu} \frac{m^t x^u}{(x_s x^s)^{3/2}} =$$

$$\epsilon^{ijk} \epsilon_{ktu} m^t \partial_j \frac{x^u}{(x_s x^s)^{3/2}} = T^i, \tag{3}$$

where we have used that m^j is a constant and the notation $(\nabla \times \mathbf{A})^i = T^i$. Using that $\epsilon^{ijk}\epsilon_{ktu} = \epsilon^{ijk}\epsilon_{tuk} = \delta^i{}_t \delta^j{}_u - \delta^i{}_u \delta^j{}_t$,

$$T^i = (\delta^i{}_t \delta^j{}_u - \delta^i{}_u \delta^j{}_t) m^t \partial_j \frac{x^u}{(x_s x^s)^{3/2}} = (\delta^i{}_t \delta^j{}_u - \delta^i{}_u \delta^j{}_t) m^t \left(\frac{\partial_j x^u}{x^3} - \frac{3}{2} x^u \frac{\partial_j (x_s x^s)}{x^5}\right. =$$

$$(\delta^i{}_t\delta^j{}_u-\delta^i{}_u\delta^j{}_t)m^t(\frac{\delta_j{}^u}{x^3}-\frac{3}{2}x^u\frac{(\partial_j x_s x^s+x_s\partial_j x^s)}{x^5}=$$

$$(\delta^i{}_t\delta^j{}_u-\delta^i{}_u\delta^j{}_t)m^t(\frac{\delta_j{}^u}{x^3}-\frac{3}{2}x^u\frac{(\partial_j g_{rs}x^r x^s+x_s\delta_j{}^s)}{x^5}=$$

$$(\delta^i{}_t\delta^j{}_u-\delta^i{}_u\delta^j{}_t)m^t(\frac{\delta_j{}^u}{x^3}-\frac{3}{2}x^u\frac{(\delta_j{}^r x_r+x_s\delta_j{}^s)}{x^5}=$$

$$(\delta^i{}_t\delta^j{}_u-\delta^i{}_u\delta^j{}_t)m^t(\frac{\delta_j{}^u}{x^3}-\frac{3}{2}x^u\frac{(x_j+x_j)}{x^5}=$$

$$(\delta^i{}_t\delta^j{}_u-\delta^i{}_u\delta^j{}_t)m^t(\frac{\delta_j{}^u}{x^3}-\frac{3x^u x_j}{x^5}=$$

$$\frac{m^i}{x^3}\delta^j{}_u\delta_j{}^u-\frac{3m^i x^j x_j}{x^5}-m^j\frac{\delta^i{}_j}{x^3}+\frac{3m^j x^i x_j}{x^5}=$$

$$3\frac{m^i}{x^3}-\frac{3m^i}{x^3}-\frac{m^i}{x^3}+\frac{3m^j x^i x_j}{x^5}=$$

$$-\frac{m^i}{x^3}+\frac{3n^i m^j n_j}{x^3}=\frac{3n^i m^j n_j-m^i}{x^3}. \tag{4}$$

We see that Eq.(4) can be written as

$$\frac{3\mathbf{n}(\mathbf{n}.\mathbf{m})-\mathbf{m}}{x^3}. \tag{5}$$

Problem 2:

a) The rank of $\epsilon^{\alpha\beta\gamma\rho}\partial_\beta F_{\gamma\rho}$ is 1 because there is only one free (non-contracted) index.

b) For $\alpha=0$ there are 6 values of $\epsilon^{0\beta\gamma\rho}$ that are non-zero: $\epsilon^{0123}=\epsilon^{0312}=\epsilon^{0231}=1$ and $\epsilon^{0213}=\epsilon^{0321}=\epsilon^{0132}=-1$, then the equation becomes:

$$\epsilon^{0123}\partial_1F_{23}+\epsilon^{0312}\partial_3F_{12}+\epsilon^{0231}\partial_2F_{31}+\epsilon^{0213}\partial_2F_{13}+\epsilon^{0321}\partial_3F_{21}+\epsilon^{0132}\partial_1F_{32}, \tag{5}$$

replacing the values of the Levi-Civita tensor components and the elements of $F_{\alpha\rho}$ by its values in terms of the field components we obtain:

$$\partial_1(-B_x)+\partial_3(-B_z)+\partial_2(-B_y)-\partial_2B_y-\partial_3B_z-\partial_1B_x=-2\nabla.\mathbf{B}=0. \tag{6}$$

c) Let's write the Lorentz transformation by components:

$$x'^0=\gamma(x^0-\beta_1x^1-\beta_2x^2-\beta_3x^3), \tag{7}$$

$$x'^i=x^i+\frac{(\gamma-1)}{\beta^2}(\beta_1x^1+\beta_2x^2+\beta_3x^3)\beta_i-\gamma\beta^i x^0, \tag{8}$$

for $i = 1$, 2, and 3. If $\mathbf{v} = c(2/3, 0, 1/3)$ then $\vec{\beta} = (\frac{2}{3}, 0, \frac{1}{3})$ with $\beta^2 = 5/9$ and $\gamma = 3/2$. Then the transformation takes the form:

$$x'^0 = \frac{3}{2}x^0 - x^1 - \frac{1}{2}x^3, \tag{9}$$

$$x'^1 = -x^0 + \frac{7}{5}x^1 + \frac{1}{5}x^3, \tag{10}$$

$$x'^2 = x^2 \tag{11}$$

$$x'^3 = -\frac{1}{2}x^0 + \frac{1}{5}x^1 + \frac{11}{10}x^3, \tag{12}$$

Then,

$$M^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{3}{2} & -1 & 0 & -\frac{1}{2} \\ -1 & \frac{7}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{5} & 0 & \frac{11}{10} \end{pmatrix}, \tag{13}$$

d) We know that

$$F'^{01} = M^0{}_\mu M^1{}_\nu F^{\mu\nu} = M^0{}_\mu M^1{}_\nu g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}. \tag{14}$$

Since only F_{03} and F_{30} are non-zero Eq.(14) only has two terms:

$$F'^{01} = M^0{}_0 M^1{}_3 g^{00} g^{33} F_{03} + M^0{}_3 M^1{}_0 g^{33} g^{00} F_{30} =$$

$$-\frac{3}{2}\frac{1}{5}E + (-\frac{1}{2})(-1)E = \frac{E}{5}. \tag{15}$$

e) Notice that $B'_y = F'_{13}$, then

$$F'_{13} = g_{1\mu} g_{3\nu} F'^{\mu\nu} = g_{11} g_{33} F'^{13} = F'^{13} = M^1{}_\alpha M^3{}_\beta F^{\alpha\beta} = M^1{}_\alpha M^3{}_\beta g^{\alpha\tau} g^{\beta\rho} F_{\tau\rho}. \tag{16}$$

Since only the (03) and (30) components of $F_{\tau\rho}$ are non-zero we obtain:

$$F'_{13} = M^1{}_0 M^3{}_3 g^{00} g^{33} F_{03} + M^1{}_3 M^3{}_0 g^{33} g^{00} F_{30} =$$

$$-\frac{1}{5}(-\frac{1}{2})(-E) + (-1)(-1)(\frac{11}{10})E = E. \tag{17}$$

Problem 3:

a) There is azimuthal symmetry so we propose:

$$\Phi^I(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{18}$$

for $0 \leq r \leq a$ because since $r = 0$ is in the region we cannot have negative powers of r. For $r \geq a$ we propose

$$\Phi^{II}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta), \tag{19}$$

since positive powers of r would diverge when $r \to \infty$ and thus, they cannot appear in the potential.

Now we use the b.c. to find A_l and B_l. We know that at $r = a$, $\Phi(a,\theta) = V_0 \sin^2\theta$. Let us write the surface potential in terms of Legendre polynomials. Notice that the potential is an even function of $\cos\theta$ since $\sin^2\theta = 1 - \cos^2\theta$. Then we only have to consider polynomials with even l. We know that $P_0(\cos\theta) = 1$ and $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$. Thus, it is clear that the potential will be a linear combination of these two polynomials. We find that

$$\sin^2\theta = \frac{2}{3}(P_0(\cos\theta) - P_2(\cos\theta)). \tag{20}$$

Then at $r = a$ we find that

$$\Phi^{II}(a,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta) = \frac{2}{3} V_0 (P_0(\cos\theta) - P_2(\cos\theta)). \tag{21}$$

Due to the orthogonality of the Legendre polynomials we know that the coefficients of each polynomial have to be the same on both sides of Eq.(21). Then we find that

$$B_l = 0, \tag{22}$$

for all values of l different from 0 and 2 and

$$B_0 = \frac{2}{3} V_0 a, \tag{23}$$

and

$$B_2 = -\frac{2}{3} V_0 a^3. \tag{24}$$

Since the potential is continuous at $r = a$ we find that $A_l = 0$ for all l different from 0 and 2 and:

$$A_0 = \frac{2}{3} V_0, \tag{25}$$

and

$$A_2 = -\frac{2}{3}\frac{V_0}{a^2}. \tag{26}$$

Then

$$\Phi^{I}(r,\theta) = \frac{2}{3} V_0 - \frac{2}{3}\frac{V_0 r^2}{a^2} P_2(\cos\theta), \tag{27}$$

and

$$\Phi^{II}(r,\theta) = \frac{2 V_0 a}{3r} - \frac{2}{3}\frac{V_0 a^3}{r^3} P_2(\cos\theta). \tag{28}$$

b)

i) The addition of a grounded shell of radius $b > a$ does not affect the potential for $r \leq a$ which remains the same, and for $r \geq b$ the potential vanishes since the potential is 0 at $r = b$. Then we need to find the potential for $a \leq r \leq b$ which will be given by:

$$\Phi^{II}(r,\theta) = \sum_{l=0}^{\infty} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(\cos\theta). \tag{29}$$

Since at $r = b$ the potential vanishes we find that

$$A_l = -\frac{B_l}{b^{2l+1}}. \tag{30}$$

Using Eq.(30) and the boundary condition at $r = a$ we find that

$$\sum_{l=0}^{\infty}[\frac{B_l}{a^{l+1}} - \frac{B_l}{b^{2l+1}}a^l]P_l(\cos\theta) = \frac{2}{3}V_0(P_0(\cos\theta) - P_2(\cos\theta)). \tag{31}$$

Comparing the coefficients of the Legendre polynomials we see that $A_l = B_l = 0$ for all l different from 0 and 2 and

$$B_0 = \frac{2}{3}\frac{abV_0}{(b-a)}, \tag{32}$$

$$A_0 = -\frac{2}{3}\frac{aV_0}{(b-a)}, \tag{33}$$

$$B_2 = -\frac{2}{3}\frac{a^3b^5V_0}{(b^5-a^5)}, \tag{34}$$

and

$$A_2 = \frac{2}{3}\frac{a^3V_0}{(b^5-a^5)}. \tag{34}$$

Then we find that

$$\Phi^{II}(r,\theta) = \frac{2}{3}\frac{aV_0}{(b-a)}(\frac{b}{r} - 1) - \frac{2}{3}\frac{a^3V_0}{(b^5-a^5)}(\frac{b^5}{r^3} - r^2)P_2(\cos\theta). \tag{35}$$

ii) As said above, the potential inside the sphere of radius a has not changed because the potential on its surface remained the same.

iii) The result of part (b) becomes the same as the result of part (a) when $b \to \infty$.