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Last time:

$W_N(n_1)$ probability of taking n_1 steps to the right after N steps.

$\langle n_1 \rangle = Np$ $\langle n_2 \rangle = Nq$

$\overline{\Delta n_1^2} = \langle n_1^2 \rangle - \langle n_1 \rangle^2 = Npq$ and $\Delta^* n_1 = \sqrt{Npq}$

Now let's find $\langle m \rangle$ and $\overline{\Delta m^2}$ for $P_N(m)$

probability of being at a distance m from the start after N steps

$m = n_1 - n_2$ $N = n_1 + n_2 \therefore m = 2n_1 - N$

$\langle m \rangle = 2\langle n_1 \rangle - N = 2Np - N = N(2p-1) =$

$= N(p + \underbrace{p-1}_{-q}) = N(p-q)$

$$\Delta m = m - \langle m \rangle = 2n_1 - N - N(p-q) = 2n_1 - N - N(2p-1) =$$

$$= 2(n_1 - Np) - N + N = 2(n_1 - \langle n_1 \rangle) = 2\Delta n_1^2$$

$$\overline{\Delta m^2} = 4\,\overline{\Delta n_1^2} = 4Npq$$

$$\Delta^* m = 2\sqrt{Npq}$$

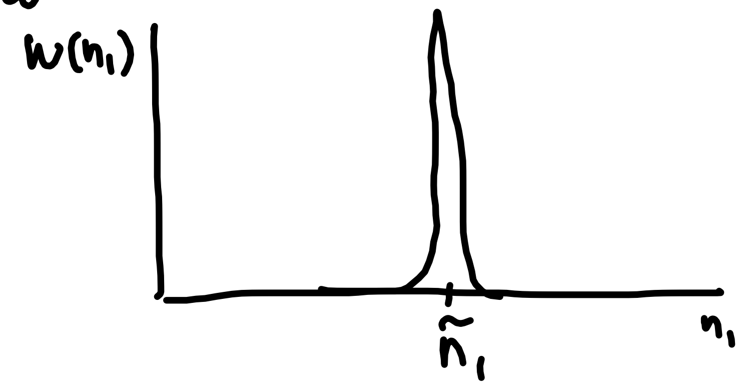
If $p = q$ then $\langle m \rangle = 0$ and $\Delta^* m = \sqrt{N}$

Large N limit of the probability distribution.

If N is very large we'll see that $W_N(n_1)$ and $P_N(m)$ become gaussian except for the case in which $p \ll 1$ or $q \ll 1$. Then you'll get a Poisson distribution that does not have a finite dispersion (see HW #1).

If $N \to \infty$ $\quad \dfrac{\Delta n_1^*}{\langle n_1 \rangle} = \sqrt{\dfrac{q}{p}}\,\dfrac{1}{\sqrt{N}}$

Now



Since $W(n_1)$ has a sharp maximum at $n_1 = \tilde{n}_1$ we can make a Taylor expansion of $W(n_1)$ about $\tilde{n}_1$.

Notice that close to $\tilde{n}_1$ $|W(n_1+1) - W(n_1)| \ll W(n_1)$

so n_1 can be considered continuous.

We know that

$$\left.\frac{\partial W}{\partial n_1}\right|_{\tilde{n}_1} = 0 \qquad \text{or} \qquad \left.\frac{d \ln W}{d n_1}\right|_{\tilde{n}_1} = 0$$

A series for $\ln W$ converges faster than for W.

$$\ln W(n_1) = \underbrace{\ln W(\tilde{n}_1)}_{\tilde{W}} + \underbrace{\frac{d\ln W}{dn_1}}_{0}\Big|_{\tilde{n}_1} \eta + \frac{1}{2} \underbrace{\frac{d^2 \ln W}{dn_1^2}}_{-|B_2|}\Big|_{\tilde{n}_1} \eta^2 + \cdots$$

Then

$$\boxed{W(n_1) \simeq \tilde{W} e^{-\frac{1}{2}|B_2|\eta^2}} \quad ①$$

Also we know that

$$W_N(n_1) = \frac{N!}{n_1!(N-n_1)!} p^{n_1} q^{N-n_1}$$

$$\ln W_N(n_1) = \ln N! - \ln n_1! - \ln (N-n_1)! + n_1 \ln p + (N-n_1) \ln q \quad ②$$

For large n, $\ln n!$ can be considered continuous.

$$\frac{d \ln n!}{dn} \simeq \frac{\ln (n+1)! - \ln (n)!}{1} = \ln\left[\frac{(n+1)!}{n!}\right] = \ln(n+1) \approx$$

$$\approx \ln n \quad \text{since } n \gg 1. \quad (3)$$

Then

$$\frac{d \ln W}{dn_1} \cong -\ln n_1 + \ln (N - n_1) + \ln p - \ln q$$

We can find $\tilde{n}_1$ since $\left.\frac{d \ln W}{dn_1}\right|_{\tilde{n}_1} = 0$

$$0 = \ln \underbrace{\frac{(N-\tilde{n}_1)}{\tilde{n}_1} \frac{p}{q}}_{1}$$

Then $N - \tilde{n}_1 = \tilde{n}_1 \frac{q}{p}$

$$N = \tilde{n}_1 \left(\frac{q}{p} + 1\right) = \frac{\tilde{n}_1}{p} \overbrace{(q+p)}^{1}$$

$$\boxed{\tilde{n}_1 = Np}$$

Also

$$B_2 = \left.\frac{d^2 \ln W}{dn_1^2}\right|_{\tilde{n}_1} = -\left.\frac{1}{n_1}\right|_{\tilde{n}_1} - \left.\frac{1}{N-n_1}\right|_{\tilde{n}_1} = -\frac{1}{Np} - \frac{1}{N-Np} =$$

$$= -\frac{1}{Np} - \frac{1}{N\underbrace{(1-p)}_{q}} = -\frac{q+p}{Npq} = -\frac{1}{Npq}$$

$$|B_2| = \frac{1}{Npq} \equiv \frac{1}{\overline{\Delta n_1^2}}$$

To find $\tilde{W}$ we request normalization of $W(n_1)$:

$$1 = \int_{-\infty}^{\infty} W(n_1)\, dn_1 \cong \int_{-\infty}^{\infty} W(\tilde{n}_1 + \eta)\, d\eta = \int_{-\infty}^{\infty} \tilde{W} e^{-\frac{|B_2|}{2}\eta^2} d\eta = 1$$

because $W(n_1) \approx 0$ for n_1 away from $\tilde{n}_1$.

We know that

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \quad \text{then}$$

$$I = \tilde{W}\sqrt{\frac{2\pi}{|B_2|}} = 1 \Rightarrow \tilde{W} = \sqrt{\frac{|B_2|}{2\pi}} = \sqrt{\frac{1}{2\pi Npq}}$$

Then

$$W(n_1) = \sqrt{\frac{1}{2\pi Nqp}}\; e^{-\frac{1}{2}\frac{(n_1 - Np)^2}{2Npq}}$$

or to remove N:

$$W(n_1) = \sqrt{\frac{1}{2\pi\,(\overline{\Delta n_1^2})}}\; e^{-\frac{1}{2}\left(\frac{n_1 - \langle n_1\rangle}{2(\overline{\Delta n_1^2})}\right)^2}$$

(4) gaussian shape.

Gaussian probability distribution

From ④ we can also obtain the large N limit of $P_N(m)$:

$$m = 2n_1 - N \Rightarrow n_1 = \frac{m+N}{2}$$

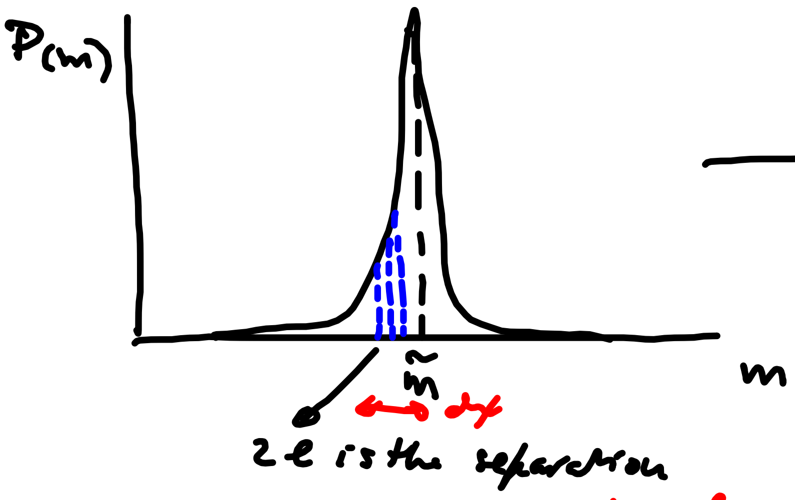
$$P(m) = W\left(\frac{N+m}{2}\right) = [2\pi Npq]^{-1/2} e^{-\frac{[m-\overbrace{N(p-q)}^{\langle m\rangle}]^2}{\underbrace{8Npq}_{2\cdot 4Npq = 2\langle \Delta m^2\rangle}}}$$

$$= [2\pi Npq]^{-1/2} e^{-\frac{m-\langle m\rangle}{2\langle \Delta m^2\rangle}}$$

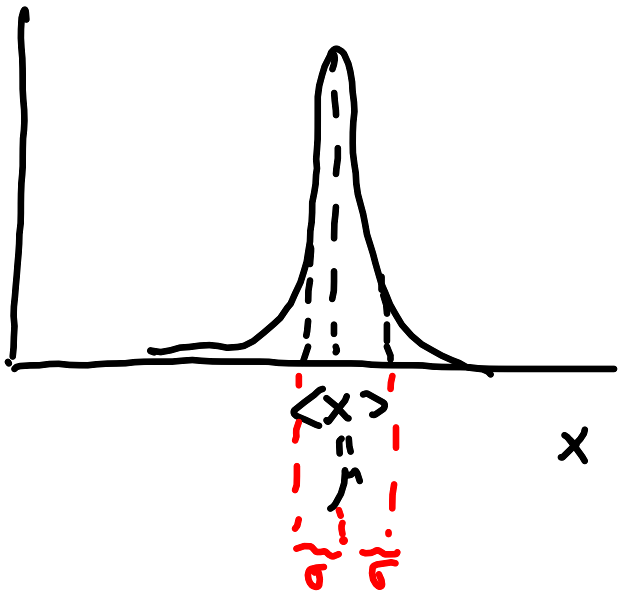
We can express $P(m)$ in terms of the actual displacement $x = m\ell$ ℓ: size of the step.

If $\ell << L$ L: length of interest such as the length of a container.

then x can be considered continuous. ℓ: of the order of the mean free path.



$\longrightarrow \mathcal{P}(x) = \frac{P(m)}{2\ell}$



in each interval dx there are $\frac{dx}{2\ell}$ values of m.

$\mu = (p-q)N\ell$

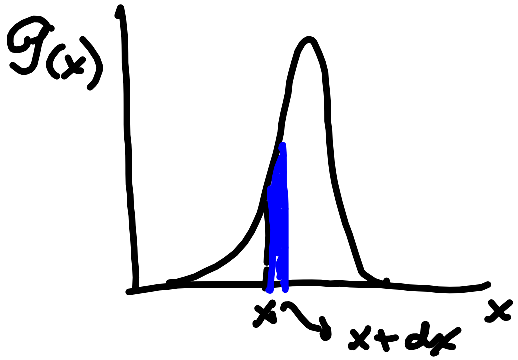
$\sigma = 2\sqrt{Npq}\,\ell$

$\mathcal{P}(x)$ is a probability density.

$$\mathcal{P}(x)\,dx = \frac{P(m)}{2\ell}\,dx = \frac{1}{2\pi\sigma}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Gaussian distribution

$\mathcal{P}(x)\,dx$ is the probability of finding the walker in the interval $(x, x+dx)$ and it is given by the blue area in the figure.



Notice that if $p \neq q$ the gaussian approximation is still valid when $N \to \infty$ as long as p is not too small. If $p \ll 1$ then $\langle \Delta n_1^2 \rangle = \sqrt{\frac{q}{p}}\sqrt{\frac{1}{N}}$ and it does not go to zero. You'll get Poisson's distribution.

Probability Distributions for more than 1 random variables: discrete case.

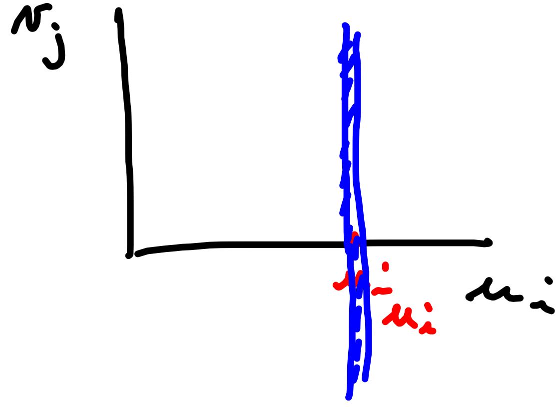
$P(u_i, v_j)$ prob that a random variable u takes the value u_i of M possible values and a random variable v takes the value v_j of N possible values, $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

- $P(u_i, v_j)$ is normalized

$$\sum_{i=1}^{M} \sum_{j=1}^{N} P(u_i, v_j) = 1$$

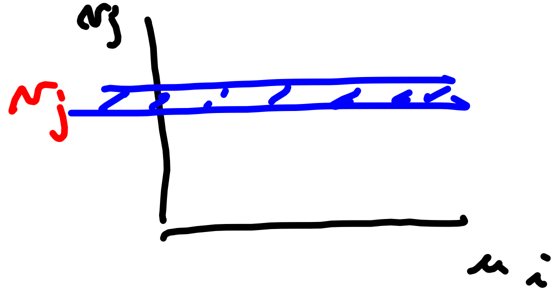
- $P_u(u_i) = \sum_{j=1}^{N} P(u_i, v_j)$ (5)

  prob of selecting u_i regardless of the value of v.



- $P_v(v_j) = \sum_{i=1}^{M} P(u_i, v_j)$

  prob of selecting v_j regardless of the value of u.



- Then

$$\sum_{i=1}^{N} P_u(u_i) \overset{(5)}{=} \sum_{i=1}^{N} \sum_{j=1}^{M} P(u_i, v_j) = 1$$

- If u and v are independent variables, or uncorrelated variables then

$$P(u_i, v_j) = P_u(u_i)\, P_v(v_j)$$

- $$\overline{F}(u,v) = \sum_{i,j} F(u_i, v_j)\, P(u_i, v_j)$$

- $$\overline{f}(u) = \sum_{i,j} f(u_i)\, P(u_i, v_j) = \sum_{i=1}^{M} f(u_i) \underbrace{\sum_{j=1}^{N} P(u_i, v_j)}_{P_u(u_i)}$$

$$\overset{5}{=} \sum_{i=1}^{M} f(u_i)\, P_u(u_i)$$

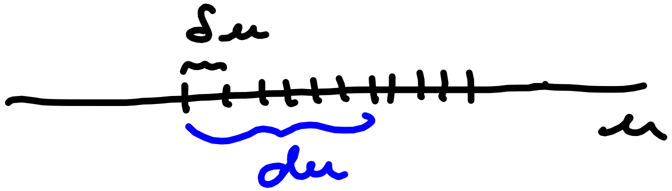
- $\overline{F + G} = \overline{F} + \overline{G}$
- $\overline{f(u)\, g(v)} = \overline{f(u)}\; \overline{g(v)}$

Probability Density
(continuous case)

If the length of interest is much larger than the interval between the values of the discrete random variables we can define a continuous probability density

$$\frac{P(u)}{\delta u}\, du = \mathcal{P}(u)\, du$$

probability density.



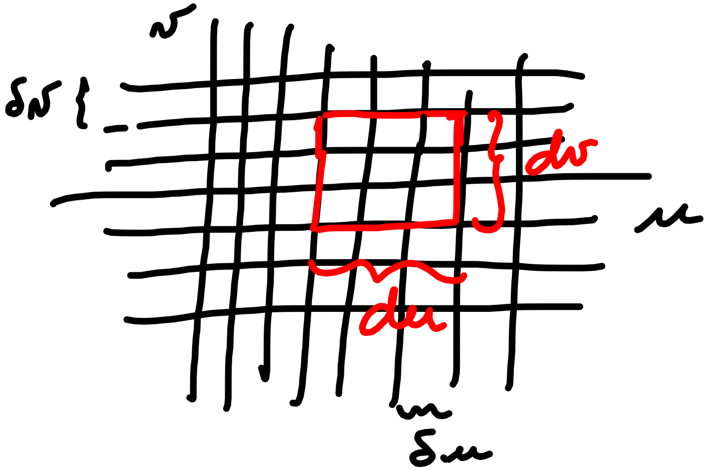
$\delta u \ll du \ll U$

Now sums become integrals:

$$\sum_{i=1}^{N} P(u_i) = 1 \longrightarrow \int_{a_1}^{a_2} \mathcal{P}(u)\, du = 1$$

$a_1 - a_2$
range of u

Beyond 1 variable:



$$\mathcal{P}(u,v)\, du\, dv = \frac{P(u,v)}{\delta u\, \delta v} du\, dv$$

probability that
u and v fall
in $dA = du\, dv$ (red area)

Now

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} du\, dv\ \mathcal{P}(u,v) = 1 \qquad \text{etc.}$$