8/27

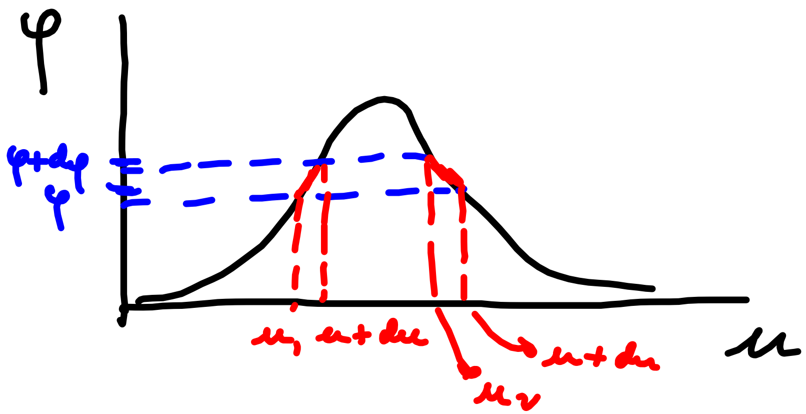
Functions of random variables

u : random variable

with $P(u)\,du$ probability of $u \leq u_i \leq u + du$

consider a continuous function $\varphi(u)$

We want to find $P(\varphi)\,d\varphi$ which gives the probability that $\varphi(u)$ will take values between φ and $\varphi + d\varphi$.



$$\mathcal{P}(\varphi)\,d\varphi = \int \mathcal{P}(u)\,du =$$

↳ summing over the values of u that provide $\varphi(u)$ between φ and $\varphi + d\varphi$

$$= \int_{\varphi}^{\varphi+d\varphi} \mathcal{P}(u) \left|\frac{du}{d\varphi}\right| d\varphi = \sum_{\{u_i\}} \mathcal{P}(u) \left|\frac{du_i}{d\varphi}\right| d\varphi$$

Central Limit Theorem

A sum of many independent and identically distributed random variables will tend to be distributed as a normal distribution if their variance is finite. In other words: there will be a peak at the average value.

Another way: The average of the averages of any distribution with finite variance will be normal.

General calculation of $\mathcal{P}(x)\,dx$

$$X = \sum_{i=1}^{N} s_i \quad (1)$$

x: total displacement in a 1D random walk in N steps.

$$\mathcal{P}(x)\,dx = \int\int \cdots \int w(s_1)\,w(s_2) \ldots w(s_N)\, ds_1\, ds_2 \ldots ds_N$$

integral over all combinations of $\{s_i\}$ that satisfy (1)

$w(s_i)$: prob. that after i steps the walker is between s_i and $s_i + ds_i$

To calculate the integrals it is easier to use a Lagrange multiplier so the integrals go from $-\infty$ to $+\infty$ unrestricted.

$$\mathcal{P}(x)\,dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w(s_1)\ldots w(s_N)\left[\delta\left(x - \sum_{i=1}^{N} s_i\right) dx\right]$$

$$ds_1 \ldots ds_N$$

But

$$\delta\left(x - \sum_{i=1}^{N} s_i\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik\left[\sum_i s_i - x\right]} dk \qquad (A.7.14)$$

$\therefore$

$$\mathcal{P}(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{-ikx} \int_{-\infty}^{\infty} ds_1\, w(s_1)\, e^{-iks_1} \cdots \int_{-\infty}^{\infty} ds_N\, w(s_N)\, e^{iks_N}$$

if $Q(k) = \int_{-\infty}^{\infty} ds\, w(s)\, e^{iks}$ characteristic function.

$$\mathcal{P}(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; Q^N(k)\, e^{-ikx} \qquad 1$$

If N is very large $Q^N(k)$ becomes very small particularly for large values of k because e^{iks} oscillates a lot inside the integral.

So we can focus on small values of ks and expand $Q(k)$ in powers of ks.

$$Q(k) = \int_{-\infty}^{\infty} ds\, w(s)\, e^{iks} = \int_{-\infty}^{\infty} ds\, w(s) \left(1 + iks - \frac{1}{2}k^2 s^2 + \cdots\right)$$

$$= 1 + ik\overline{s} - \frac{1}{2}k^2\,\overline{s^2} + \cdots.$$

$$\overline{s^n} = \int_{-\infty}^{\infty} ds\, w(s)\, s^n$$ is the n moment of $w(s)$.

We will assume that $|w(s)| \to 0$ if $|s| \to \infty$ (the distribution has finite width).

$$\therefore \quad \ln Q^N(k) = N \ln Q(k) = N \ln [1 + \underbrace{ik\bar{s} - \tfrac{1}{2}k^2\overline{s^2} + \dots}_{y}]$$

if $y \ll 1$ $\quad \ln(1+y) = y - \frac{1}{2}y^2 \dots$

$$= N[i\bar{s}k - \tfrac{1}{2}\overline{s^2}k^2 - \tfrac{1}{2}(i\bar{s}k)^2 + \dots] =$$

$$= N[i\bar{s}k - \tfrac{1}{2}(\underbrace{\overline{s^2} - \bar{s}^2}_{\overline{\Delta s^2} = \langle s^2\rangle_c})k^2 + \dots]$$

← expanded in terms of $\langle s^n\rangle_c$ cumulants

$$\therefore \quad Q^N(k) \simeq e^{iN\bar{s}k - \frac{1}{2}N\overline{(\Delta s)^2}k^2} \qquad (2)$$

Plugging ② in ①:

$$\mathcal{P}(x) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-i(N\bar{s}-x)k - \frac{1}{2}N(\Delta s)^2 k^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ gaussian form

A.4.2

$\mu \equiv N\bar{s}$

$\sigma^2 = N\overline{(\Delta s)^2}$

This demonstrates the central limit theorem for $w(s)$.

Cumulants:

$$\langle x \rangle_c = \bar{x}$$

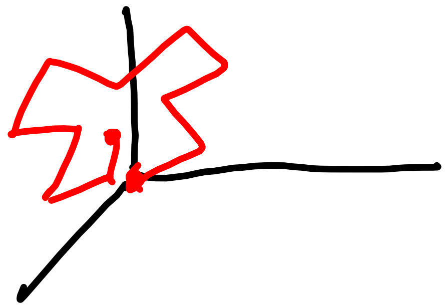
$$\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x^3 \rangle_c = \langle x^3 \rangle - 3 \langle x^2 \rangle \langle x \rangle + 2 \langle x \rangle^3$$

$$\vdots$$

These will be useful for diagramatic expansions.

Example: Problem 1.28: Random walk in 3D



$w(\bar{s})\,d^3s$ probability of the displacement $\bar{s}$ being between $\bar{s}$ and $\bar{s}+d^3s$

$\mathcal{P}(\bar{r})\,d^3r = ?$ probability of being between $\bar{r}$ and $\bar{r}+d^3r$ after N steps.

$$\mathcal{P}(\bar{r})\,d^3r = \underbrace{\iint\cdots\int}_{\text{restricted by the constraint}} w(\bar{s}_1)\cdots w(\bar{s}_N)\,d^3s_1\cdots d^3s_N$$

$$\bar{r} = \sum_{i=1}^{N} \bar{s}_i$$

we will use the Lagrange multiplier to lift the restriction on the $\int$.

$$\mathcal{P}(\bar{r})\,d^3r = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w(\bar{s}_1)\cdots w(\bar{s}_N)\left[\delta\left(\bar{r}-\sum_{i=1}^{N}\bar{s}_i\right)\right]d^3r \; ds_1\cdots ds_N$$

but $\delta\left(\bar{r}-\sum_{i=1}^{N}\bar{s}_i\right) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\, e^{i\bar{k}\cdot\left[\sum_{i=1}^{N}\bar{s}_i-\bar{r}\right]}$

$\therefore$

$$\mathcal{P}(\bar{r})\,d^3r = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\, e^{-i\bar{k}\cdot\bar{r}} \prod_{i=1}^{N} \underbrace{\int_{-\infty}^{\infty} w(s_i)\, e^{i\bar{k}\cdot\bar{s}_i}\, d^3s_i}_{Q(\bar{k})}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k\, e^{-i\bar{k}\cdot\bar{r}}\, Q^N(k)$$

Notice that

$$Q(\bar{k}) = \int_{-\infty}^{\infty} d^3s \; w(\bar{s})\, e^{-i\bar{k}\cdot\bar{s}} = \int_{-\infty}^{\infty} d^3s \, w(\bar{s}) \sum_{j=0}^{\infty} \frac{(i\bar{k}\cdot\bar{s})^j}{j!} =$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \int_{-\infty}^{\infty} (i)^j w(\bar{s}) \, (k_a s^a)^j \, d^3s =$$

$$= i\bar{k} \cdot \underbrace{\int_{-\infty}^{\infty} w(\bar{s})\, \bar{s}\, d^3s}_{\langle \bar{s} \rangle} - \frac{1}{2} k_a k_b \underbrace{\int_{-\infty}^{\infty} w(\bar{s})\, s^a s^b d^3s}_{\langle S^2_{ab} \rangle}$$

Description of Systems of Many Particles.

- Specify the state of a system:

  * For 3 dice the 3 numbers on top of each dice after a throw specify the state.

  * For a system of $\underset{N}{\wedge}$ particles:
    - classical system: $\bar{q}_i$ $i = 1, \cdots, N$, $\bar{p}_i$
    - we need to provide $6N$ numbers to specify the state of the system. in 3D.
    - $f = 3N$ # of degrees of freedom.
    - you need to provide $2f$ values to specify the state.

- For a quantum mechanical system characterized by a wave function $\psi(q_1, \ldots, q_f)$

f = # of degrees of freedom = # of quantum numbers.

Classical description: useful for systems at high temperature and diluted.

1D f particles $q_1, \ldots, q_f$ position

$p_1, \ldots p_f$ momentum

$H(\{q_i\}, \{p_i\}, t)$ Hamiltonian

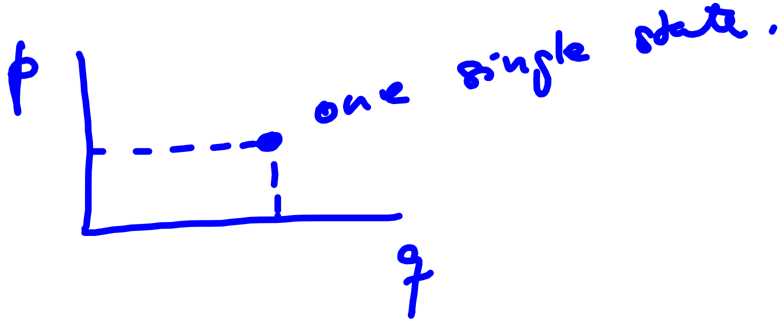
$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$q_i = q_i(q_{0,i}, t)$$
$$p_i = p_i(p_{0,i}, t)$$

$\{q_i, p_i\}$ define a phase space that has $2f$ dimensions,

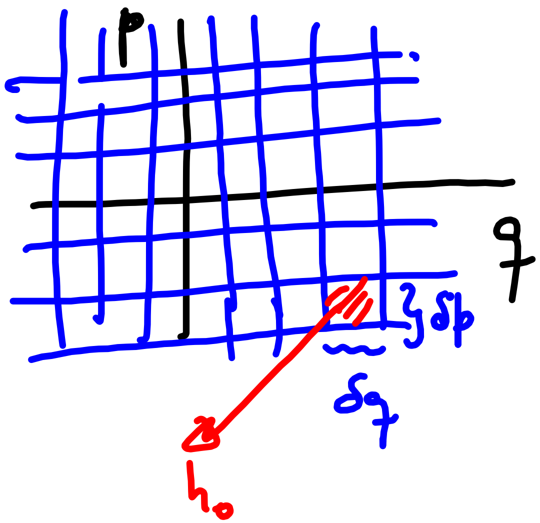
Example: 1 particle in 1D:

$\{p, q\}$ dimension of phase space $= 2f = 2$



Since we do not know p and q exactly we will divide phase space in small cells of size

$$h_0 = \delta h = \delta q\, \delta p$$



You give a number r to each cell and the state of the system is identified by this number.

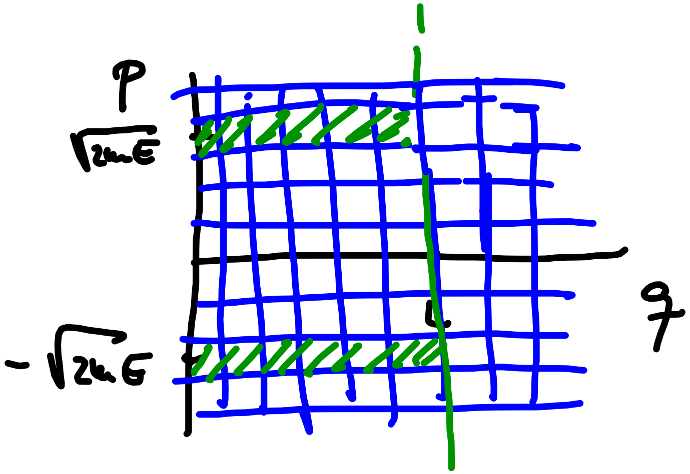
Now several values of p and q within δp and δq correspond to the same state.

Consider that we know something about the macroscopic state of our system:

1) the particle is in a box of length L.

2) The particle has energy $E = \frac{p^2}{2m}$.

Then $0 \le q \le L$ and $p = \pm\sqrt{2mE}$



The green cells represent the only accessible states.

If we had N particles in 3D we will have 6N dimensions in phase space.

$\{q_1, \dots, q_f, p_1 \dots p_f\}$ $f = 3N$ $2f$ coordinates.

$h_0^f = \delta q_1 \delta p_1 \dots \delta q_f \delta p_f$ is the hypervolume of the cell.

Microstate: microscopic state of a systems.

It is labeled by r which indicates a cell in phase space for the classical case or a set of quantum numbers for the quantum case.

If we have 2 spins with two possible spin quantum numbers each:

$\uparrow\uparrow \quad \uparrow\downarrow \quad \downarrow\uparrow \quad \downarrow\downarrow$

$r = 1 \quad 2 \quad 3 \quad 4$ for possible microstates.