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Effects of collisions in the function $f(\bar{r}, \bar{v}, t)$.

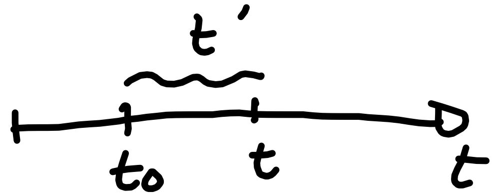
Path integral formalism.

We know that

$$\underbrace{\mathcal{P}(t)\,dt}_{} = e^{-t/\tau}\frac{dt}{\tau}$$

probability that a particle suffers a collision at time t after being undisturbed from time 0.

Define $t_0 = t - t'$ ①



and $t_0 - dt' \overset{①}{=} t - t' - dt' = t - (t' + dt')$

Probability of suffering a collision at time $t_0 - dt'$ and continue undisturbed after the time t':

$$\mathcal{P}(t')\,dt' = \frac{e^{-t'/\tau}\,dt'}{\tau}$$

$f^{(0)}(r_0, v_0, t_0)\, d^3r_0\, d^3v_0$: # of molecules around $\bar{r}_0$ with velocity $\sim \bar{v}_0$ that suffered a collision at time t_0.

Assumptions: · the effect of collisions is to restore equilibrium, i.e., M–B velocity distribution.

$$f^{(0)}(\bar{r}_0, \bar{v}_0, t_0) = n\left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\frac{1}{2}\beta m(\bar{v}_0 - \bar{u})^2}$$

$\bar{u}$: mean velocity of the molecules.

We saw that without collisions all the molecules $f^{(0)}(\bar{r}_0, \bar{v}_0, t_0)\, d^3r_0\, d^3v_0$ would arrive to $\bar{r}$ with $\bar{v}$ at time t, but due to the collisions only a fraction arrives:

functions of t'

$$f(\bar{r},\bar{v},\underline{\underline{t}}) = \int_0^\infty f^{(0)}(\bar{r}_0,\bar{v}_0,t-t')\, e^{-t'/\tau}\,\frac{dt'}{\tau} \qquad (2)$$

all the molecules that up to time t suffered collisions and then moved without colliding to reach $\bar{r}, \bar{v}$ at time t.

the molecules are locally in M-B equilibrium

probability of not colliding at $t = t'$ after having collided at $t = \tau_1$.

Notice that (2) can be integrated by parts.

$$f(\bar{r},\bar{v},t) = \int_0^\infty f^{(0)} e^{-t'/\tau} \frac{dt'}{\tau} = -f^{(0)} e^{-t'/\tau}\Big|_0^\infty -$$

$$u = f^{(0)} \qquad \int_0^\infty -\frac{df^{(0)}}{dt'} e^{-t'/\tau} dt' =$$

$$u' = \frac{df^{(0)}}{dt'}$$

$$v' = \frac{e^{-t'/\tau}}{\tau} \qquad = f^{(0)} + \int_0^\infty \frac{df^{(0)}}{dt'} e^{-t'/\tau} dt'$$

$$v = -e^{-t'/\tau}$$

$$\Delta f = f(\bar{r},\bar{v},t) - f^{(0)}(\bar{r},\bar{v},t) = \int_0^\infty \frac{df^{(0)}}{dt'} e^{-t'/\tau} dt' \qquad (3)$$

Electrical conductivity:

Before we assumed n = fixed over all the gas (which is not true out of equilibrium) and we used $\langle v_z \rangle$. Now we are going to consider the local version of n given by $f(\bar{r}, \bar{v}, t)$.

$\bar{E} = (0, 0, E_z)$ E-field.

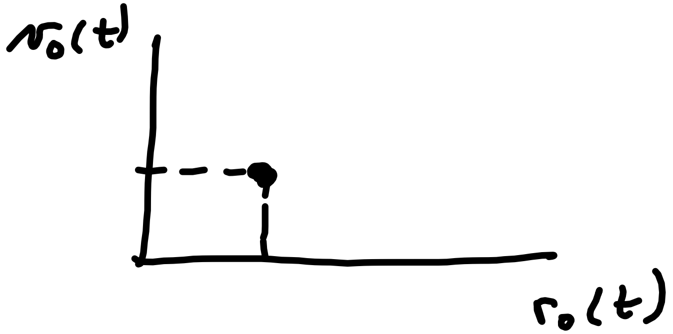
- Dilute gas
- Charged particles.
- mass m.

$f^0(\vec{r}, \vec{v}, t) = g(\varepsilon)$ $\quad \varepsilon = \frac{1}{2} m v^2$ local equilibrium distribution.

$g(\varepsilon) = n \left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\beta \varepsilon}$ Maxwell-Boltzmann's form

$n = N/V$

At time t consider



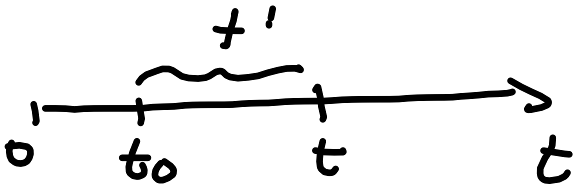
what is $r(t_0)$, $v(t_0)$, etc.?

$$\frac{dv_x}{dt_0} = 0$$ no acceleration along x

$$\frac{dv_y}{dt_0} = 0$$ " " " y.

$$m\frac{dv_z}{dt_0} = eE$$ acceleration along z.

Let's $t' = t - t_0$



$$\frac{df^{(0)}}{dt'} = \frac{df^{(0)}}{dt_0}\underbrace{\frac{dt_0}{dt'}}_{-1} = -\frac{df^{(0)}}{dt_0} = -\frac{\partial g}{\partial v_z}\frac{\partial v_z}{\partial t_0} =$$

$$= -\frac{\partial g}{\partial v_z}\frac{eE}{m} = -\frac{eE}{\cancel{m}}\cancel{m}\, v_z\frac{\partial g}{\partial \varepsilon} = -eE v_z\frac{\partial g}{\partial \varepsilon}$$

$$\mathcal{E} = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

$$\frac{\partial g}{\partial v_z} = \frac{\partial g}{\partial \mathcal{E}} \frac{\partial \mathcal{E}}{\partial v_z} = \frac{\partial g}{\partial \mathcal{E}} \cancel{\frac{1}{2}} \cancel{2} m v_z = m v_z \frac{\partial g}{\partial \mathcal{E}}$$

If v_z does not change in $t = \tau$ because $\mathcal{E}$ is small $\frac{\partial v_z}{\partial t} \tau = \frac{F}{m} \tau = \frac{e\mathcal{E}}{m} \tau << \langle v \rangle$

$\therefore \frac{dg}{\partial \mathcal{E}}(t) \approx \frac{dg}{d\mathcal{E}}(t_0)$ and $v_z(t) \approx v_z(t_0)$

$\therefore$

$$\Delta f = \int_0^{\infty} \frac{df^{(0)}}{dt'} e^{-t'/\tau} dt' = -e \mathcal{E} v_z \frac{\partial g}{\partial \varepsilon} \int e^{-t'/\tau} dt'$$

or since $\Delta f = f(\bar{r}, \bar{v}, t) - f^{(0)}(\bar{r}, \bar{v}, t)$

$$\Rightarrow f(\bar{r}, \bar{v}, t) = \Delta f + \underbrace{f^{(0)}(\bar{r}, \bar{v}, t)}_{g(\varepsilon)}$$

$$f(\bar{r}, \bar{v}, t) = g(\varepsilon) - e\mathcal{E} v_z \frac{\partial g}{\partial \varepsilon} \underbrace{\int_0^\infty e^{-t'/\tau} dt'}_{-\tau e^{-t'/\tau}\big|_0^\infty = \tau} =$$

$$= -e\mathcal{E} v_z \tau \frac{\partial g}{\partial \varepsilon} + g(\varepsilon)$$

The density of current j_x is given by

$$j_u = e \int d^3 \bar{v}\, f\, v_u \quad (5) \qquad v_u = \bar{v} \cdot \hat{u}$$

The difference with what we did before $j_u = e n \langle v_z \rangle$ is that now "$\langle v_z \rangle$" is calculated with a position and velocity dependent distribution f while before we used a uniform M-B.

In equilibrium $f^{(0)} = g(\varepsilon)$

$j_u = 0$ since (5) has an odd integrand

v_u : odd $g(\varepsilon)$: even.

But $j_z \neq 0$ if $E \neq 0$ and $j_z \parallel E$ $\therefore$

$$\sigma_{ee} = \frac{j_z}{E} = \frac{e\int d^3v\, f\, v_n}{E} =$$

$$= e\,\overbrace{\frac{\int d^3v\, g(\varepsilon)\, v_n}{E}}^{0} - \frac{\tau e^2 \cancel{E}}{\cancel{E}} \int v_z \frac{\partial g}{\partial \varepsilon} v_n\, d^3v =$$

$$= -\tau e^2 \int d^3v \frac{\partial g}{\partial \varepsilon} v_z^2 = \tau e^2 \beta \underbrace{\int d^3v\, g\, v_z^2}_{n\langle v_z^2\rangle} =$$

$$g = n\left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\beta\varepsilon}$$

$$\frac{\partial g}{\partial \varepsilon} = -\beta g$$

$$\cancel{\tfrac{1}{2}} m\langle v_z^2\rangle = \cancel{\tfrac{1}{2}} kT$$

$$\langle v_z^2\rangle = \frac{1}{m\beta}$$

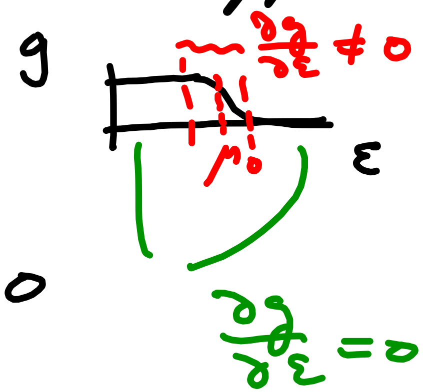
$$= \tau e^2 \beta \frac{1n}{m\beta} =$$

$$= \frac{\tau e^2 n}{m} \sim$$

$$\sigma_{el} = \frac{e^2 n}{m} \bar{\tau}$$

in agreement with what we obtained before.

However, this approach allows us to calculate σ_{el} for electrons in a metal (degenerate Fermi gas) something that we could not do with the previous approach.



Fermi-Dirac:

$$g(\varepsilon) \propto \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

$$\frac{dg(\varepsilon)}{d\varepsilon} \neq 0$$

only close to $\varepsilon = \mu_0$.

Then only the electrons very close to the Fermi surface participate in electrical conduction. In the expression for σ τ is replaced by τ_F.

$$\boxed{\sigma_{el} = -e^2 \tau_F \int d^3r \frac{\partial g}{\partial \varepsilon} v_z^2 = \frac{n e^2}{m} \tau_F}$$