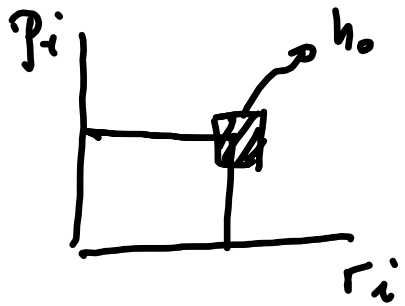
10/1

Last time:

We found the number of molecules in an ideal gas in a cell h_0 in phase space characterized by momentum between $\bar{p}$ and $\bar{p}+d\bar{p}$ and position between $\bar{r}$ and $\bar{r}+d\bar{r}$.

N: total # of molecules



We obtained:

$$N P(\bar{r},\bar{p})\, d^3\bar{r}\, d^3\bar{p} \quad (1)$$

with

$$P(\bar{r},\bar{p}) = \tilde{C}\, e^{-\frac{\beta p^2}{2m}} \quad (2)$$

Distribution of velocities:

Let's write ① and ② in terms of $\bar{v}$ instead of $\bar{p}$ knowing that $\bar{p} = m\bar{v}$.

We define:

$f(\bar{r},\bar{v})\, d^3\bar{r}\, d^3\bar{v}$: average number of molecules with the center of mass at $\bar{r} - \bar{r}+d\bar{r}$ and velocity between $\bar{v}$ and $\bar{v}+d\bar{v}$.

using ① and ②

$$f(\bar{r},\bar{v})\, d^3\bar{r}\, d^3\bar{v} = C\, e^{-\beta \frac{v^2 m}{2}}\, d^3r\, d^3v \quad ③$$

Notice that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3v \int_0^L\int_0^L\int_0^L d^3r \; f(\bar{r},\bar{v}) = N \quad (4)$$

(total # of molecules) in the gas.

Then we can find C in (3) using (4):

$v_x^2+v_y^2+v_z^2 = \sum_{i=1}^{3} v_i^2$

$$N = C \iiint_{-\infty}^{\infty} \underbrace{\iiint_0^L e^{-\frac{\beta v^2 m}{2}} d^3r}_{V} \, d^3v = C V \underbrace{\left[\int_{-\infty}^{\infty} e^{-\frac{\beta v_i^2 m}{2}} dv_i\right]^3}_{\left(\frac{2\pi}{\beta m}\right)^{1/2}} =$$

integral table

$$= C V \left(\frac{2\pi}{\beta m}\right)^{3/2}$$

Then

$$C = \frac{N}{V}\left(\frac{\beta m}{2\pi}\right)^{3/2} \equiv n \left(\frac{\beta m}{2\pi}\right)^{3/2}$$

$n = N/V$ density of particles.

Then

$$f(\bar{r},\bar{v})\, d^3r\, d^3v = n\left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-\frac{\beta v^2 m}{2}}\, d^3r\, d^3v =$$

$$= n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{v^2 m}{2kT}}\, d^3r\, d^3v = \underbrace{f(\bar{v})\, d^3v}_{\text{velocity distribution}}\, d^3r$$

Then

$$f(\bar{v})\, d^3v = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{v^2 m}{2kT}}\, d^3v \qquad (5)$$

It indicates the # of molecules per unit volume with velocity between $\bar{v}$ and $\bar{v}+d^3v$

Then

$$\iiint_{-\infty}^{\infty} f(\bar{v})\, d^3v = n \quad \left(\begin{matrix}\text{density of}\\ \text{particles}\end{matrix}\right)$$

N/V

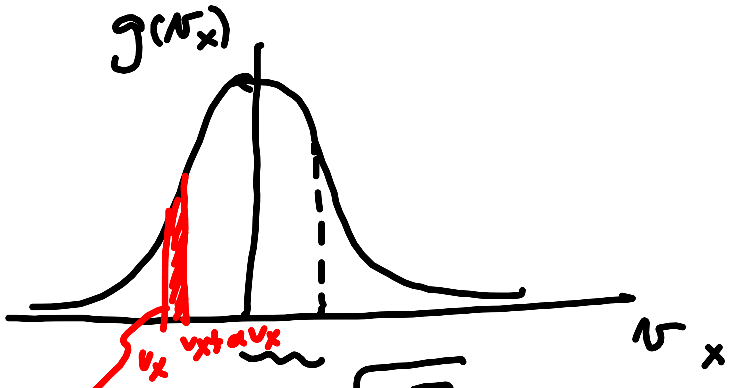
Other distributions: $g(v_x)\,dv_x$

# of molecules per unit volume with x component of the velocity between v_x and $v_x + dv_x$. We can integrate $f(\bar{v})\,d^3v$ (5) over v_y and v_z so:

$$g(v_x)\,dv_x = \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y\, f(\bar{v})\,dv_x =$$

$$= n\left(\frac{m}{2\pi kT}\right)^{3/2} dv_x\, e^{-\frac{mv_x^2}{2kT}} \underbrace{\left[\int_{-\infty}^{\infty} dv_i\, e^{-\frac{mv_i^2}{2kT}}\right]^2}_{\left(\frac{2\pi kT}{m}\right)^{1/2}} =$$

$$= n\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_x^2}{2kT}}\,dv_x$$



$\sigma = \sqrt{\frac{kT}{m}}$

\# of molecules per unit volume with v_x between v_x and $v_x + dv_x$

Gaussian shape

Notice that

$$\int_{-\infty}^{\infty} g(v_x)\, dv_x = n$$

$$\langle v_x^k \rangle = 0 \quad \text{if } k \text{ is odd.}$$

$$\langle v_x^2 \rangle = \frac{1}{n} \int_{-\infty}^{\infty} v_x^2\, g(v_x)\, dv_x = \frac{kT}{m}$$

use table

From equipartition theorem we know that

$$\frac{1}{2} m v_x^2 = \frac{kT}{2} \Rightarrow \langle v_x^2 \rangle = \frac{kT}{m}$$

We knew that $\varepsilon = \frac{3}{2} kT$ (per molecule)

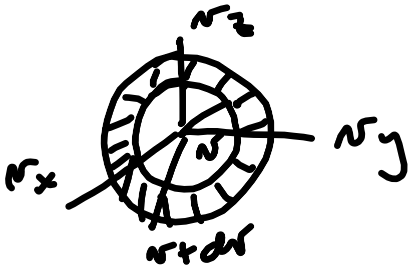
So we see that the energy is equally distributed among v_x, v_y and v_z.

x

<u>Speed distribution</u>:

Let's define: $F(v)\,dv$ → # of molecules per unit volume with speeds between v and $v+dv$

$v = |\bar{v}|$



$$F(v)\,dv = \int_v^{v+dv}\int_0^{\pi}\int_0^{2\pi} f(\bar{v})\,\underbrace{v^2 \sin\theta\, dv\, d\theta\, d\varphi}_{d^3v}$$

Then

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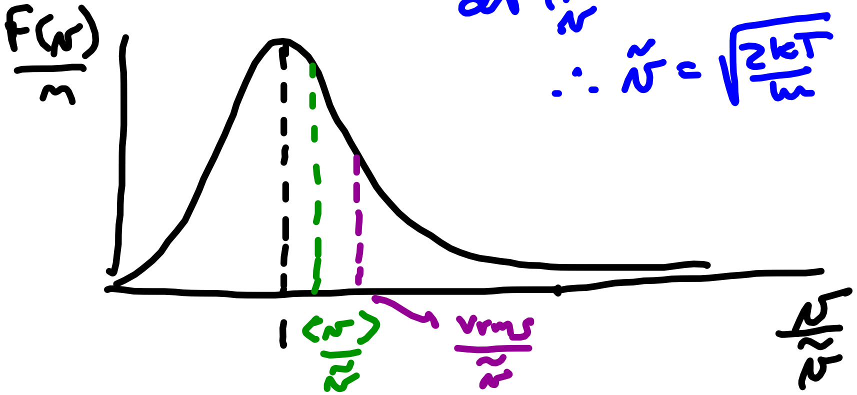
$$F(v) = 4\pi f(v)\, v^2\, dv = 4\pi m \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}} dv$$

increasing with v decreasing with v

$$0 = \left.\frac{dF}{dv}\right|_{\tilde{v}} = 2v e^{-\frac{mv^2}{2kT}} + v^2\left(-\frac{m}{kT}v\right)e^{(\,)}$$

$$\therefore \tilde{v} = \sqrt{\frac{2kT}{m}}$$



It will have a maximum at $v = \tilde{v} > 0$.

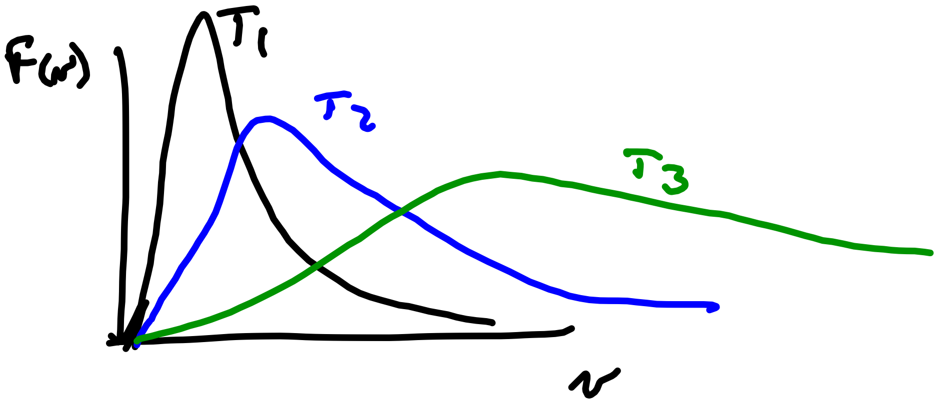
$$\int_0^\infty F(v)\, dv = m.$$

$$\langle v \rangle = \frac{1}{m}\int_0^\infty v\, F(v)\, dv = \sqrt{\frac{8}{\pi}\frac{kT}{m}}$$

$$\langle v^2 \rangle = \frac{1}{m}\int_0^\infty v^2 F(v)\, dv = \frac{3kT}{m}$$

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3kT}{m}}$$

We see from the expressions of $\tilde{v}$ that

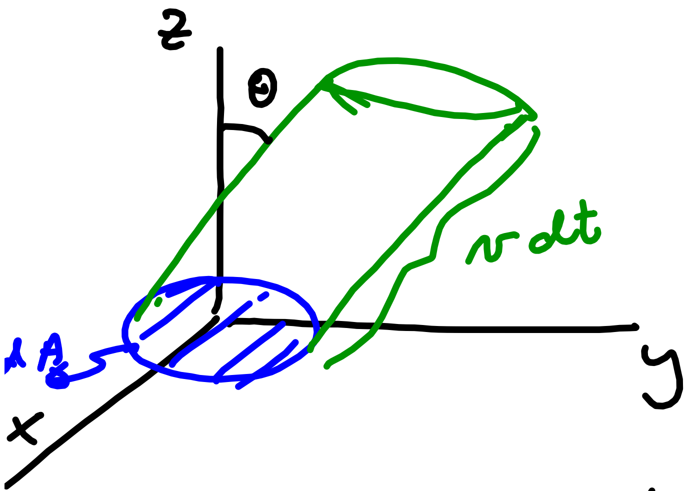


$T_1 < T_2 < T_3$

The distribution becomes broader as T increases.

$\tilde{v}$ increases with T.

Number of molecules striking a surface:



All the molecules inside the cylinder with v in v–$v+dv$ interval will impact the surface dA during the time interval dt

Density of molecules in the cylinder with velocity v – $v+dv$ is

$$f(\bar{v})\, d^3\bar{v}\, \underbrace{dA\, v\, dt\, \cos\theta}_{dV}$$

# of molecules that hit the surface with direction θ.

Per unit area and per unit time the # of particles that hit the surface with direction θ is:

$$\phi(\bar{v}) = f(\bar{v})\, d^3v\, v\cos\theta \quad (1)$$

flux if $\theta = 0$ (particles come $\perp$ to dA) and ϕ is maximum

$\phi = 0$ if $\theta = \frac{\pi}{2}$.

Flux: # of particles that strike the wall per unit area and unit time in all directions.

$$\phi_0 = \int_{v_z>0} dv_z \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\, f(\bar{v})\, v\cos\theta = \int$$

spherical coordinates

$$\boxed{\Phi_0} = \int_0^\infty v^3 f(v)\,dv \underbrace{\int_0^{\pi/2} \sin\theta \underbrace{\cos\theta\, d\theta}_{d(\sin\theta)}}_{\frac{1}{2}\sin^2\theta \Big|_0^{\frac{\pi}{2}}} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} =$$

$$= \pi \int_0^\infty v^3 f(v)\,dv = \frac{\overset{\frac{1}{2}}{\pi}\, n \bar{v}}{4\pi} = \boxed{\frac{n\bar{v}}{4}}$$

But

$$\bar{v} = \frac{1}{n}\int d^3v\, f(v)\, v = \frac{4\pi}{n} \underbrace{\int_0^\infty f(v)\, v^3\, dv}_{\frac{n\bar{v}}{4\pi}}$$

Then

$$\phi_0 = \frac{n\bar{v}}{4} = \frac{n}{4}\sqrt{\frac{8kT}{\pi m}} = \frac{n k T}{4 k T}\sqrt{\frac{8kT}{m\pi}} = \bar{p}\sqrt{\frac{8kT}{16k^2T^2 m\pi}}$$

$$\boxed{= \frac{\bar{p}}{\sqrt{2\pi m k T}}}$$

$\bar{p} = n k T$ (eq. of state for ideal gas)