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Last time:
we were studying $Z_{BE}$

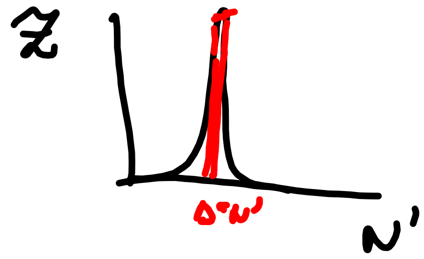
with $\sum_r n_r = N$. (1)

But we found that in order to deal with (1) it was better to calculate $\mathcal{Z}_{BE}$.

We found that

$$\mathcal{Z}_{B.E} = \sum_{N'=0}^{\infty} Z(N') e^{-\alpha N'} \cong$$

$$\mathcal{Z} \simeq Z(N)\, e^{-\alpha N}\, \Delta^* N'$$




$\Delta^* N' \ll N$

$$\ln \mathcal{Z} = \ln Z(N) - \alpha N + \cancel{\ln \Delta^* N'}$$

$$\therefore \ln Z(N) = \ln \mathcal{Z} + \alpha N \qquad (2)$$

But

$$\mathcal{Z} = \sum_R e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \dots)}\, e^{-\alpha(n_1 + n_2 + \dots)} =$$

$$= \sum_{n_1, n_2, \dots} e^{-(\alpha+\beta\varepsilon_1)n_1} e^{-(\alpha+\beta\varepsilon_2)n_2} \dots =$$

$$= \underbrace{\sum_{n_1=0}^{\infty} e^{-(\alpha+\beta\varepsilon_1)n_1}}_{\frac{1}{1-e^{-(\alpha+\beta\varepsilon_1)}}} \underbrace{\sum_{n_2=0}^{\infty} e^{-(\alpha+\beta\varepsilon_2)n_2}}_{\frac{1}{1-e^{-(\alpha+\beta\varepsilon_2)}}} \dots =$$

$$= \prod_r \frac{1}{1-e^{-(\alpha+\beta\varepsilon_r)}} = \mathcal{Z}$$

$$\ln \mathcal{Z} = -\sum_r \ln\left(1-e^{-\alpha-\beta\varepsilon_r}\right) \quad (3)$$

Replacing ③ in ② we obtain:

$$\ln Z(N) = \alpha N - \sum_r \ln\left(1 - e^{-\alpha - \beta \varepsilon_r}\right) \quad (4)$$

We do not yet know $\alpha$, but we know that $\ln \mathcal{Z}$ has a maximum at $N' = N$ then

$$0 = \left.\frac{\partial \ln \mathcal{Z}}{\partial N'}\right|_{N'=N} \overset{②}{=} \left.\frac{\partial}{\partial N'}\left[\ln Z(N') - \alpha N'\right]\right|_{N'=N}$$

$$= \left.\frac{\partial \ln Z(N')}{\partial N'}\right|_{N'=N} - \alpha = 0 \quad (5)$$

From (5) we see that

$$\alpha = \alpha(N)$$

since $\alpha$ is the parameter that forces $Z(N')$ to peak at $N' \equiv N$.

Plugging (2) into (5):

depends on $N$ only through $\alpha$

$$\frac{\partial}{\partial N}\left[\ln Z + \alpha N\right] - \alpha = 0$$

$$\frac{\partial}{\partial N}\left[\alpha(N)\, N + \ln Z\right] - \alpha = 0$$

$$\frac{\partial}{\partial N}\left[\alpha(N)\,N + \ln \mathcal{Z}\right] - \alpha = 0$$

$$\cancel{\alpha} + N\frac{\partial \alpha}{\partial N} + \frac{\partial \ln \mathcal{Z}}{\partial \alpha}\frac{\partial \alpha}{\partial N} \cancel{- \alpha} = 0$$

$$\underbrace{\left(N + \frac{\partial \ln \mathcal{Z}}{\partial \alpha}\right)}_{0}\frac{\partial \alpha}{\partial N} = 0$$

$\neq 0$ since $\alpha = \alpha(N)$

$$0 \overset{3}{=} N + \frac{\partial}{\partial \alpha}\left[-\sum_r \ln\left(1 - e^{-\alpha - \beta \varepsilon_r}\right)\right] =$$

$$= N - \sum_r \frac{e^{-\alpha-\beta\varepsilon_r}}{1 - e^{-\alpha-\beta\varepsilon_r}} = N - \sum_r \frac{1}{e^{\alpha+\beta\varepsilon_r} - 1}$$

Then

$$N = \sum_r \frac{1}{e^{\alpha+\beta\varepsilon_r}-1} \qquad (6)$$

Also we know that

$$\frac{\partial \ln Z}{\partial \alpha} = 0$$ since $Z$ is independent of $\alpha$.

Also:

$$\langle n_s \rangle = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_s} \overset{(4)}{=} -\frac{1}{\beta}\frac{\partial}{\partial \varepsilon_s}\Big[\underbrace{\alpha N - \sum_r \ln\left(1-e^{-\alpha-\beta\varepsilon_r}\right)}_{\ln Z}\Big]$$

$$= -\frac{1}{\beta}\left[\cancel{N\frac{\partial \alpha}{\partial \varepsilon_s}} - \frac{-(-\beta)e^{-\alpha-\beta\varepsilon_s}}{1-e^{-\alpha-\beta\varepsilon_s}} + \cancel{\underbrace{\frac{\partial \ln Z}{\partial \alpha}}_{-N\ (3)}\frac{\partial \alpha}{\partial \varepsilon_s}}\right]$$

Then

$$\boxed{\langle n_s \rangle = \frac{1}{e^{\alpha+\beta\varepsilon_s}-1}}$$

Bose - Einstein distribution.

Notice that $F = F(T, V, N)$

$$F = -kT \ln Z$$

$$dF = -S\,dT - P\,dV + \mu\,dN$$

$$\Rightarrow \left.\frac{\partial F}{\partial N}\right|_{T,V} = \mu \quad (7)$$

Then

$$\left.\frac{\partial F}{\partial N}\right|_{T,V} = -kT \underbrace{\left.\frac{\partial \ln Z}{\partial N}\right|_{T,V}}_{\alpha} = -kT\alpha \quad (8)$$

Comparing (7) with (8) we found :

$$\alpha = -\frac{\mu}{kT} = -\beta\mu$$

$\therefore$ $\mu < 0$ for B-E systems.

$\therefore$

$$\langle n_s \rangle = \frac{1}{e^{\beta(\varepsilon_s - \mu)} - 1}$$

Bose-Einstein

$\beta \to \infty \Rightarrow T \to 0$

we need $\varepsilon_s - \mu > 0$ which means that if $\varepsilon_s = 0$ $\mu < 0$.

Notice that photons have $s = 1$ these are bosons and for them we found $\langle n_s \rangle = \frac{1}{e^{\beta\varepsilon_s} - 1}$ they have $\alpha = \mu = 0$.

Dispersion of $\langle n_s \rangle$:

$$\langle (\Delta n_s)^2 \rangle = \langle (n_s - \bar{n}_s)^2 \rangle = \langle n_s^2 \rangle - \langle n_s \rangle^2$$

$$\langle n_s \rangle = \frac{\sum_R n_s e^{-\beta(n_1 \varepsilon_1 + n_2 \varepsilon_2 + \cdots)}}{Z} = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \varepsilon_s} \quad (*)$$

$$\langle n_s^2 \rangle = \frac{\sum_R n_s^2 e^{-\beta(n_1 \varepsilon_1 + \cdots)}}{Z} = +\frac{1}{\beta^2 Z} \frac{\partial^2 Z}{\partial \varepsilon_s^2} \quad (8)$$

But

$$\frac{\partial}{\partial \varepsilon_s}\left(\frac{1}{Z}\frac{\partial Z}{\partial \varepsilon_s}\right) = -\frac{1}{Z^2}\left(\frac{\partial Z}{\partial \varepsilon_s}\right)^2 + \frac{1}{Z}\frac{\partial^2 Z}{\partial \varepsilon_s^2}$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial \varepsilon_s^2} = \frac{\partial}{\partial \varepsilon_s}\left(\frac{1}{Z}\frac{\partial Z}{\partial \varepsilon_s}\right) + \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \varepsilon_s}\right)^2 \quad (9)$$

Plugging (9) in (8):

$$\langle m_s^2 \rangle = \frac{1}{\beta^2}\left[\frac{\partial}{\partial \varepsilon_s}\underbrace{\left(\frac{1}{Z}\frac{\partial Z}{\partial \varepsilon_s}\right)}_{\frac{\partial \ln Z}{\partial \varepsilon_s}} + \underbrace{\frac{1}{Z^2}\left(\frac{\partial Z}{\partial \varepsilon_s}\right)^2}_{\left(\frac{\partial \ln Z}{\partial \varepsilon_s}\right)^2}\right] =$$

$$= \frac{1}{\beta^2}\left[\frac{\partial^2 \ln Z}{\partial \varepsilon_s^2} + \beta^2 \langle m_s \rangle^2\right] =$$

$$= \frac{1}{\beta^2}\frac{\partial^2 \ln Z}{\partial \varepsilon_s^2} + \langle m_s \rangle^2 \qquad (10)$$

Then $\boxed{\langle (\Delta m_s)^2 \rangle} = \langle m_s^2 \rangle - \langle m_s \rangle^2 = \boxed{\frac{1}{\beta^2}\frac{\partial^2 \ln Z}{\partial \varepsilon_s^2}}$

~~Then using 12 for B.G:~~

$$\langle (\Delta m_S^2) \rangle = -\frac{1}{\beta}$$

Combining (*) with (10) we found that

$$\langle (\Delta n_s)^2 \rangle = \frac{1}{\beta^2} \frac{\partial^2 \ln Z}{\partial \varepsilon_s^2} = -\frac{1}{\beta} \frac{\partial \langle n_s \rangle}{\partial \varepsilon_s} \qquad (11)$$

(*) $\langle n_s \rangle = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \varepsilon_s}$

For B.E.:

$$\langle (\Delta n_s)^2 \rangle = -\frac{1}{\beta} \frac{\partial \langle n_s \rangle}{\partial \varepsilon_s} = -\frac{1}{\beta} \frac{\partial}{\partial \varepsilon_s} \left[ \frac{1}{e^{\alpha+\beta\varepsilon_s}-1} \right] =$$

$$= \frac{1}{\beta} \frac{\overbrace{e^{\alpha+\beta\varepsilon_s}}^{e^{\alpha+\beta\varepsilon_s}-1+1}}{(e^{\alpha+\beta\varepsilon_s}-1)^2} \left( \frac{\partial \alpha}{\partial \varepsilon_s} + \beta \right) = \frac{1}{\beta} \left[ \underbrace{\frac{1}{e^{\alpha+\beta\varepsilon_s}-1}}_{\langle n_s \rangle} + \underbrace{\frac{1}{(e^{\alpha+\beta\varepsilon_s}-1)^2}}_{\langle n_s \rangle^2} \right] \left( \frac{\partial \alpha}{\partial \varepsilon_s} + \beta \right) =$$

$$= \left(\bar{n}_s + (\bar{n}_s)^2\right)\left(\underbrace{\frac{1}{\beta}\frac{\partial\alpha}{\partial\varepsilon_s}}_{\text{negligible}} + \cancel{\frac{\beta}{\beta}}\right) \approx$$

$$\approx \bar{n}_s\left(1 + \bar{n}_s\right)$$

Then

$$\frac{\overline{(\Delta n_s^2)}}{(\bar{n}_s)^2} = \frac{1}{\bar{n}_s} + 1$$

Dispersion for Bose-Einstein

# Fermi-Dirac distribution.

- Particles are fermions.
- $N$ particles
- $n_i = 0$ or $1$

We will write the grand-canonical partition function:

$$\mathcal{Z} = \sum_{n_1, n_2, \dots} e^{-\beta(n_1 \varepsilon_1 + n_2 \varepsilon_2 + \dots)} e^{-\alpha(n_1 + n_2 + \dots)} =$$

$$= \sum_{n_1=0}^{1} e^{-(\beta \varepsilon_1 + \alpha) n_1} \sum_{n_2=0}^{1} e^{-(\beta \varepsilon_2 + \alpha) n_2} \dots =$$

$$= \left(1+e^{-(\beta\varepsilon_1+\alpha)}\right)\left(1+e^{-(\beta\varepsilon_2+\alpha)}\right)\cdots\cdots = \prod_r \left(1+e^{-(\beta\varepsilon_r+\alpha)}\right)$$

$$\therefore \ln \mathcal{Z} = \sum_r \ln\left(1+e^{-(\alpha+\beta\varepsilon_r)}\right) \quad (1)$$

we found before that

$$\mathcal{Z} = \sum_{N'=0}^{\infty} Z(N')\, e^{-\alpha N'}$$

$$\therefore \ln \mathcal{Z} \simeq -\alpha N + \ln Z(N)$$

$$\therefore \ln Z(N) = \ln \mathcal{Z} + \alpha N \overset{(1)}{=}$$

$$\ln Z(N) = \sum_r \ln\left(1+e^{-(\alpha+\beta\varepsilon_r)}\right) + \alpha N \quad (2)$$

Since $Z(N)$ is not a function of $\alpha$ we know that

$$0 = \frac{\partial \ln Z(N)}{\partial \alpha} \overset{2}{=} \sum_r \frac{e^{-\beta\varepsilon_r-\alpha}}{1+e^{-\beta\varepsilon_r-\alpha}} + N$$

Then

$$\boxed{N} = \sum_r \frac{e^{-\beta\varepsilon_r-\alpha}}{1+e^{-\beta\varepsilon_r-\alpha}} = \boxed{\sum_r \frac{1}{e^{\beta\varepsilon_r+\alpha}+1}}$$

Since $Z = \sum_R e^{-\beta(n_1\varepsilon_1+\cdots)} \Rightarrow \langle n_s \rangle = \frac{\sum_R n_s e^{-\beta\varepsilon_1 n_1 + \cdots}}{Z}$

$$= -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_s}$$

Since $\ln Z = \sum_r \ln(1 + e^{-\beta \varepsilon_r - \alpha}) + \alpha N$

$$\langle n_s \rangle = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \varepsilon_s} = -\frac{1}{\beta} \frac{-\beta e^{-\beta \varepsilon_s - \alpha}}{1 + e^{-\beta \varepsilon_s - \alpha}} =$$

$$= \frac{e^{-\beta \varepsilon_s - \alpha}}{1 + e^{-\beta \varepsilon_s - \alpha}} = \boxed{\frac{1}{e^{\beta \varepsilon_s + \alpha} + 1}}$$

Fermi-Dirac distribution.

Notice that $0 \leq \langle n_s \rangle \leq 1$

and $\alpha = -\beta\mu$

$$\langle n_s \rangle = \frac{1}{e^{\beta(\varepsilon_s - \mu)} + 1}$$

If $\varepsilon_s$ very large $\bar{n}_s \to 0$

also $e^{\beta(\varepsilon_s - \mu)} + 1 \geq 1$

$\langle n_s \rangle \leq 1$.