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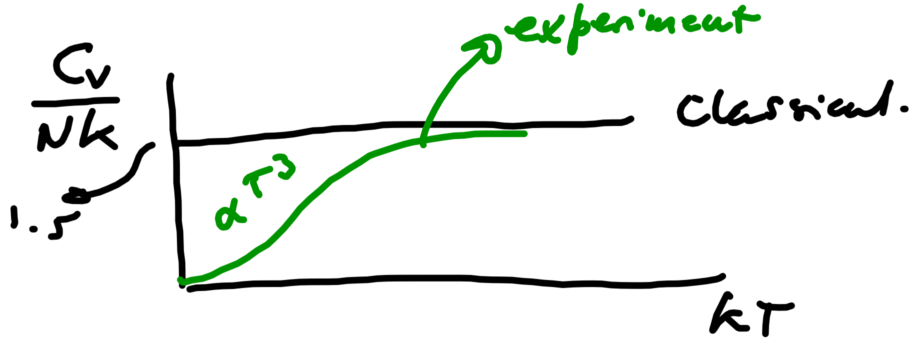
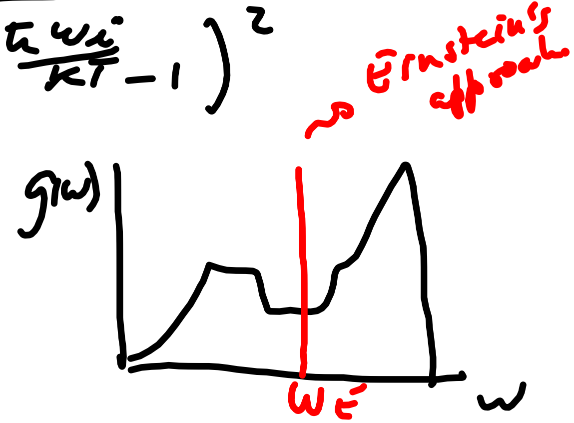
Last time:

Vibrations of the atoms in a solid: phonons.

We found:

$$C_V = \frac{\partial U}{\partial T}\bigg|_V = k \sum_{i=1}^{3N} \frac{\left(\frac{\hbar\omega_i}{kT}\right)^2 e^{\frac{\hbar\omega_i}{kT}}}{\left(e^{\frac{\hbar\omega_i}{kT}}-1\right)^2}$$





Finding $g(\omega)$ is the hardest part of the problem.

Einstein's approach: assume that $\omega_i = \omega_E$ $\forall i$.

$$c_V(T) = 3Nk \frac{(\hbar\omega_E/kT)^2 e^{\hbar\omega_E/kT}}{(e^{\hbar\omega_E/kT}-1)^2} =$$

$$= 3Nk\, E(x)$$

with $x = \frac{\hbar\omega_E}{kT} = \frac{\theta_E}{T}$ → Einstein's temperature $\quad \theta_E = \frac{\hbar\omega_E}{k}$

and $\quad \mathcal{E}(x) = \dfrac{x^2 e^x}{(e^x - 1)^2}$

When $x << 1 \Rightarrow$ high T:

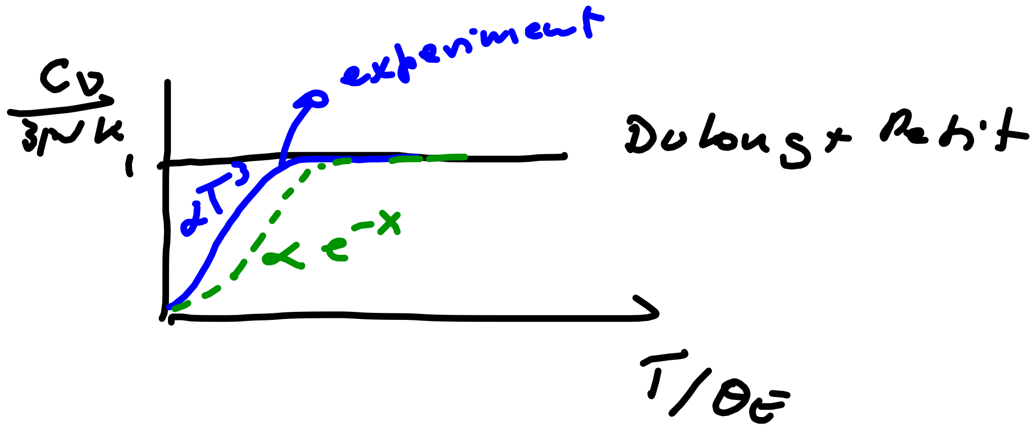
$$\mathcal{E}(x) \simeq \frac{x^2 \, 1}{(1 - x - 1)^2} = \frac{x^2}{x^2} = 1$$

$\therefore \; C_V \to 3Nk \quad$ (classical value).

When $x >> 1 \Rightarrow T \to 0$

$$\mathcal{E}(x) \simeq \frac{x^2 e^x}{(e^x)^2} \sim x^2 e^{-x}$$

$\therefore \quad C_V \sim 3Nk\, x^2 e^{-x} \longrightarrow 0$ but too fast.



Debye's model:

he considered a continuous spectrum of frequencies.

the requested fact

$$\int_0^{\omega_D} g(\omega)\, d\omega = 3N$$

because there are 3 N normal modes.

Roughest approach:

Same $g(\omega)$ as for photons but considering 3 instead of 2 propagation modes:

$$g(\omega)\, d\omega = 3 \frac{V}{2\pi^2 c_s^3} \omega^2\, d\omega$$

c_s: speed of sound.

More refined: take into account that

$c_\ell \neq c_t$ for speed of sound.

c_ℓ: longitudinal

c_t: transverse.

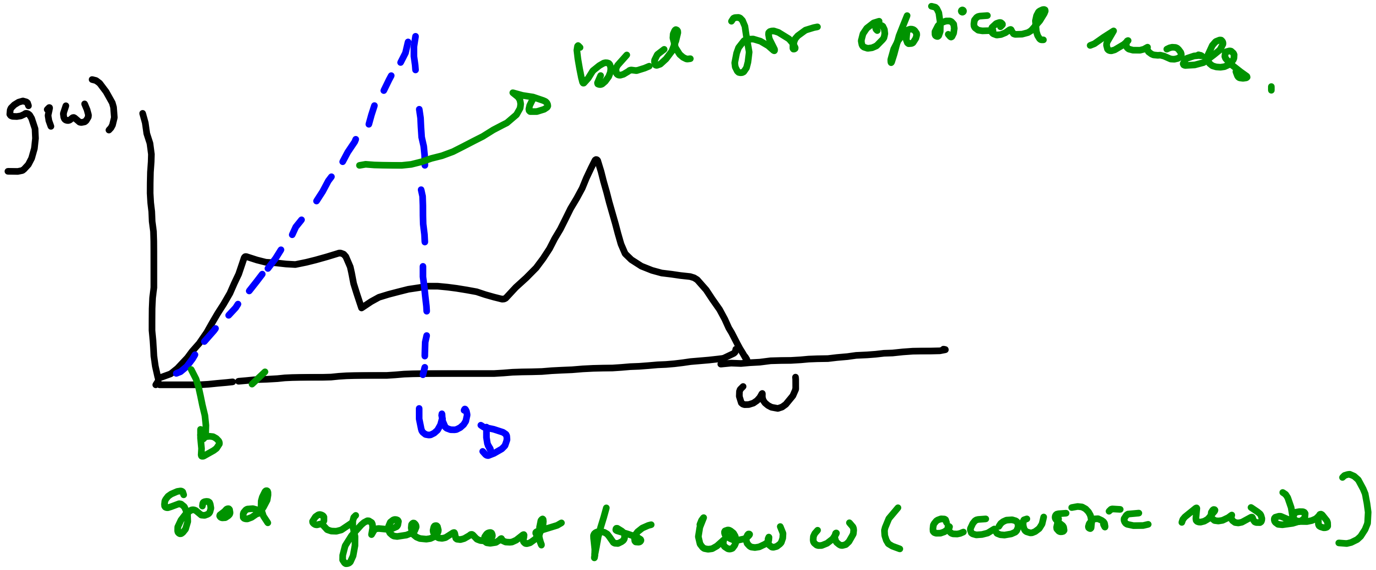
$$\int_0^{\omega_D} V\left(\frac{\omega^2}{2\pi^2 c_\ell^3} + \frac{2\,\omega^2}{2\pi^2 c_t^3}\right) d\omega = 3N$$

$$\frac{1}{3}\frac{V}{2\pi^2}\left(\frac{1}{c_\ell^3} + \frac{2}{c_t^3}\right)\omega_D^3 = 3N \quad \therefore$$

$$\omega_D^3 = 18\pi^2 \frac{N}{V}\left(\frac{1}{c_\ell^3} + \frac{2}{c_t^3}\right)^{-1}$$

Then

$$g(\omega) = \begin{cases} \dfrac{9N\omega^2}{\omega_D^3} & \text{for } \omega \le \omega_D \\ 0 & \text{for } \omega > \omega_D \end{cases}$$



You can do better calculating ω_D^L and ω_D^T:

$$\int_0^{\omega_D^L} \frac{V\,\omega^2}{2\pi^2 c_L^3}\,d\omega = N \quad \text{and} \quad \int_0^{\omega_D^T} \frac{V\,\omega^2}{\pi^2 c_T^3}\,d\omega = 2N$$

Then:

$$c_V = k\int_0^{D} \frac{g(\omega)\left(\frac{\hbar\omega}{kT}\right)^2 e^{\hbar\omega/kT}}{\left(e^{\hbar\omega/kT}-1\right)^2}\,d\omega =$$

$$= \frac{k\,9N}{\omega_D^3}\int_0^{\omega_D} \frac{\omega^2\left(\frac{\hbar\omega}{kT}\right)^2 e^{\hbar\omega/kT}}{\left(e^{\hbar\omega/kT}-1\right)^2}\,d\omega =$$

$$x = \frac{\hbar\omega}{kT} \qquad x_D = \frac{\hbar\omega_D}{kT} = \frac{\Theta_D}{T} \qquad \Theta_D = \frac{\hbar\omega_D}{k}$$

$\to$ Debye's temp.

$$C_V = \frac{9kN}{\omega_D^3}\int_0^{x_D} \frac{(kT)^3}{\hbar^3}\frac{x^2x^2e^x}{(e^x-1)^2}\,dx =$$

$$= \frac{9kN}{\omega_D^3}\frac{(kT)^3}{\hbar^3}\int_0^{x_D}\frac{x^4e^x}{(e^x-1)^2}\,dx =$$

$$= 3kN\;\frac{3}{x_D^3}\int_0^{x_D}\frac{x^4e^x}{(e^x-1)^2}\,dx = 3kN\,D(x_D)$$

$$D(x_D) = \frac{3}{x_D^3}\int_0^{x_D}\frac{x^4e^x}{(e^x-1)^2}\,dx$$

If $x_D << 1 \Rightarrow$ large T

$$D(x_D) = \frac{3}{x_D^3}\int_0^{x_D} \frac{x^4}{(1-x-1)^2}\,dx = \frac{3}{x_D^3}\int_0^{x_D} x^2\,dx =$$

$$= \frac{3}{x_D^3}\,\frac{x_D^3}{3} = 1$$

$\therefore\ C_V \to 3kN$ (classical limit).

If $x_D >> 1 \Rightarrow$ low T

$$D(x_D) \cong \frac{3}{x_D^3}\int_0^{\infty} \frac{x^4 e^x}{(e^x-1)^2} = \frac{3}{x_D^3}\,\frac{4\pi^4}{5}$$

Then

$$C_V \rightarrow \frac{12\pi^4}{5} N k \left(\frac{T}{\Theta_D}\right)^3 \propto T^3$$ as in the experiments.

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Ideal Fermi Systems:

Thermodynamics: $\quad z = e^{\mu/kT}$

$$\frac{PV}{kT} = \ln \mathcal{Z} = \sum_{\varepsilon_s} \ln\left(1 + z e^{-\beta \varepsilon_s}\right) \quad ①$$

$$N = \sum_{\varepsilon_s} \langle n_{\varepsilon_s} \rangle = \sum_{\varepsilon_s} \frac{1}{z^{-1} e^{\beta \varepsilon_s} + 1} \quad ②$$

From ② we see that $0 \le z \le \infty$ there are no the same limitations for z than in B.E. ∴

Notice that ε_0 only can have 1 or 0 fermions ∴ nothing like B.E. condensation as $T \to 0$.

Let's work with ① and ② replacing

$$\sum_{\varepsilon_0} \longrightarrow \int a(\varepsilon)\, d\varepsilon \quad \text{as for B.E.}$$

$$\frac{PV}{KT} = \int_0^\infty g\, a(\varepsilon)\, d\varepsilon \ \ln\left(1 + z e^{-\beta\varepsilon}\right) =$$

$$a(\varepsilon) = \frac{2\pi V}{h^3} (2m)^{3/2} \varepsilon^{1/2}$$

$g = 2$ for particles with $s = 1/2$

$$= \int_0^\infty g \frac{2\pi V}{h^3} (2m)^{3/2} \varepsilon^{1/2} \ln\left(1 + z e^{-\beta\varepsilon}\right) d\varepsilon =$$

One has to integrate by parts as with the bosons and we obtain:

$$\frac{P}{kT} = \frac{g}{\lambda^3} f_{5/2}(z) \qquad (1)$$

Doing the same for (2) we obtain:

$$\frac{N}{V} = \frac{g}{\lambda^3} f_{3/2}(z) \qquad (2)$$

with $\lambda = \frac{h}{(2\pi m k T)^{1/2}}$

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}\,dx}{z^{-1}e^x + 1} = z - \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} \cdots$$

good for $z \ll 1$.

Now we can obtain the thermodynamic properties:

$$U = -\frac{\partial \ln \mathcal{Z}}{\partial \beta}\bigg|_{z,V} = kT^2 \frac{\partial}{\partial T} \ln \mathcal{Z}\bigg|_{z,V} =$$

(1)

$$= \frac{3}{2} kT \frac{g\,V}{\lambda^3} f_{5/2}(z) = \frac{3}{2} PV \Rightarrow$$

(*)

(1) P/kT

$$P = \frac{2}{3}\frac{U}{V}$$

also

$$U = \frac{3}{2} NkT \frac{f_{5/2}(z)}{f_{3/2}(z)}$$

using in (*) that

$$V = \frac{N\lambda^3}{g f_{3/2}(z)} \quad (2)$$

Now

$$c_V = \frac{\partial U}{\partial T}\Big|_{N,V} = \frac{3}{2} Nk \frac{f_{5/2}(z)}{f_{3/2}(z)} + \frac{3}{2} NkT \left\{ \frac{f'_{5/2}(z)}{f_{3/2}(z)} - \right.$$

$$\left. - \frac{f_{5/2}\, f'_{3/2}}{f^2_{3/2}} \right)$$

Using:

$$\frac{1}{z} \frac{\partial z}{\partial T}\Big|_{v} = -\frac{3}{2T} \frac{f_{3/2}(z)}{f_{1/2}(z)}$$

$$v = \frac{V}{N}$$

$$\frac{c_V}{Nk} = \frac{15}{4} \frac{f_{5/2}(z)}{4 f_{3/2}(z)} - \frac{9}{4} \frac{f_{3/2}(z)}{f_{1/2}(z)}$$

$$F = N\mu - PV = NkT\left\{\ln z - \frac{f_{5/2}(z)}{f_{3/2}(z)}\right\}$$

$$S = \frac{U-F}{T} = Nk\left\{\frac{5}{2}\frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z\right\}$$

To find the properties in terms of $n = \frac{N}{V}$ we need to know $z = z(n, T)$ from

$$\frac{N}{V} = n = \frac{g}{\lambda^3} f_{3/2}(z)$$

Now let's consider the high and low T limits:

for high T and n small (ideal gas limit).

$$f_{3/2}(z) \simeq z = e^{\mu/kT} \simeq \frac{n\lambda^3}{g} = \frac{n h^3}{g\,(2\pi m kT)^{3/2}} << 1$$

from series expansion of $f_\nu(z)$

$n = \frac{g}{\lambda^3} f_{3/2}(z)$

condition for non-degenerate gas.

Then in this case $z << 1 \Rightarrow \mu$ large and negative.

$\rightarrow$ ideal gas limit.

Then

$$P = \frac{NkT}{V} \qquad U = \frac{3}{2} NkT \qquad C_V = \frac{3}{2} Nk$$

If $z \ll 1$ but not too small we can do a virial expansion as we did for B. G.:

$$\frac{PV}{NkT} = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} a_\ell \left(\frac{\lambda^3}{g v}\right)^{\ell-1}$$

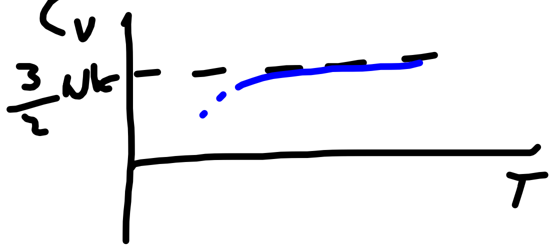
a_ℓ are the same virial coefficients obtained for B. G.

Using the virial expansion:

$$C_V = \frac{3}{2} N k \sum_{l=1}^{\infty} (-1)^{l-1} \frac{(5-3l)}{2} a_l \left(\frac{\lambda^3}{g v}\right)^{l-1} =$$

$$= \frac{3}{2} N k \left[1 - 0.0884 \left(\frac{\lambda^3}{g v}\right) + \cdots \right]$$

$C_V < \frac{3}{2} N k$ different than for B. E. in which $C_V > \frac{3}{2} N k$



Now let's consider the low T behavior:

If $\frac{n\lambda^3}{g} >> 1$ degenerate case.

$f_\nu(z)$ can be expanded in terms of $(\ln z)^{-1}$!

For $\frac{n\lambda^3}{g} \to \infty$ the expansions assume a

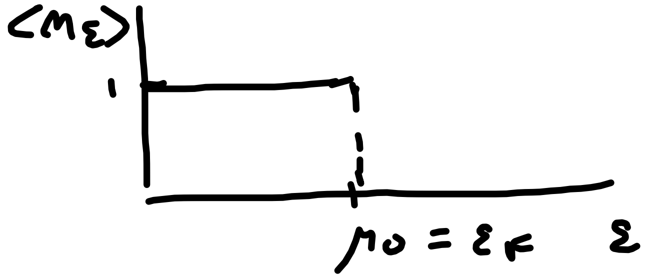
close form.

Completely degenerate gas: $T = 0$. $\frac{n\lambda^3}{g} \to \infty$

$\therefore$

$$\langle n_\varepsilon \rangle = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1} = \begin{cases} 1 & \text{if } \varepsilon < \mu_0 \\ 0 & \text{if } \varepsilon > \mu_0 \end{cases}$$

μ_0: chemical potential at $T = 0$



$\mu_0 = \varepsilon_F$: Fermi energy

$\varepsilon_F = \frac{p_F^2}{2m}$ $\quad p_F$: Fermi momentum.

Notice that

$$\int_0^{\varepsilon_F} a(\varepsilon)\, d\varepsilon = \frac{g V 4\pi}{h^3} \int_0^{\varepsilon_F} p^2 \frac{dp}{d\varepsilon}\, d\varepsilon = N =$$

$$= \int_0^{\infty} \frac{a(\varepsilon)\, d\varepsilon}{e^{\frac{(\varepsilon-\mu)}{kT}} + 1} \qquad \varepsilon_F = p_F^2/2m$$

$$\therefore \quad N = \frac{g V 4\pi}{3 h^3} p_F^3 \Rightarrow p_F = \left(\frac{3N}{4\pi g V}\right)^{1/3} h$$

$$\therefore \quad \varepsilon_F = \frac{p_F^2}{2m} = \left(\frac{6\pi^2 n}{g}\right)^{2/3} \frac{\hbar^2}{2m}$$

Ground state energy E_0

$$E_0 = \int_0^{p_F} \frac{p^2}{2m}\, a(p)\, dp = \frac{4\pi g V}{h^3} \int_0^{p_F} \frac{p^2}{2m}\, p^2\, dp =$$

$$= \frac{2\pi g V}{2m h^3}\, p_F^5$$

$$\therefore \frac{E_0}{N} = \frac{3 p_F^2}{10 m} = \frac{3}{5} \varepsilon_F \qquad \text{very high energy.}$$

$$P_0 = \frac{2}{3} \frac{E_0}{V} = \frac{2}{5} n \varepsilon_F = \left(\frac{6\pi^2}{g}\right)^{2/3} \frac{\hbar^2}{5m} n^{5/3}$$

$$\propto n^{5/3}.$$