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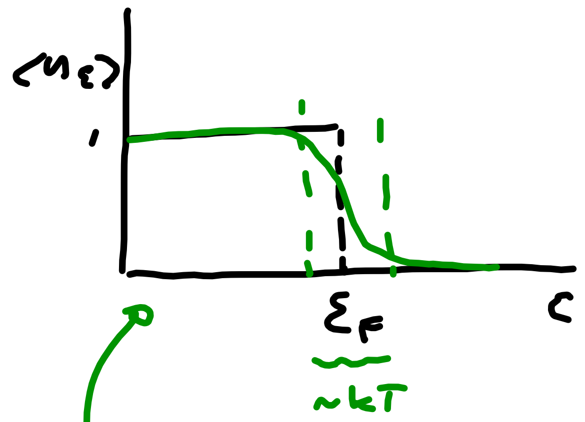
Ideal Fermi systems:

Last time:

$T=0$



If $T \gtrsim 0$

$$\langle n_\varepsilon \rangle = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}$$

Only fermions very close to ε_F become "active".

In this regime $z = e^{\mu/kT}$ is very large but not infinity.

Now $f_\nu(z)$ have to be expanded in terms of $\ln(z)$ rather than as powers of z - See Appendix E in the book.

$$f_{5/2}(z) = \frac{8}{15\pi^{1/2}} (\ln z)^{5/2} \left[1 + \frac{5\pi^2}{8} (\ln z)^{-2} + \cdots\right]$$

$$f_{3/2}(z) = \frac{4}{3\pi^{1/2}} (\ln z)^{3/2} \left[1 + \frac{\pi^2}{8} (\ln z)^{-2} + \cdots\right]$$

$$f_{1/2}(z) = \frac{2}{\pi^{1/2}} (\ln z)^{1/2} \left[1 - \frac{\pi^2}{24} (\ln z)^{-2} + \cdots \right]$$

Chemical potential:

$\mu = \mu(T)$

For $T = 0$ $\mu = \mu_0 = \varepsilon_F > 0$

For $T \to \infty$ $\mu \to$ large and negative as for the ideal gas.

We will find $\mu(T)$ for $T \gtrsim 0$

Let's start with:

$$\frac{N}{V} = \frac{g}{\lambda^3} f_{3/2}(z) \simeq \frac{g}{\lambda^3} \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left[1 + \frac{\pi^2}{8} (\ln z)^{-2} + \dots\right]$$

Eq (2) (last time)

replacing $\lambda = \frac{h}{(2\pi m kT)^{1/2}}$

$$= \frac{4\pi g}{3} \left(\frac{2m}{h^2}\right)^{3/2} (\underbrace{kT \ln z}_{\mu})^{3/2} \left[1 + \frac{\pi^2}{8} (\ln z)^{-2} + \dots\right] \quad (1)$$

zeroth order (we expect to find $\mu = \mu_0 = \varepsilon_F$):

$$z = e^{\mu/kT} \Rightarrow \mu = kT \ln z$$

Let's solve for μ in Eq. ① keeping only the first term [1] in the expansion:

$$\mu = kT \ln z \approx \left(\frac{3}{4\pi g}\frac{N}{V}\right)^{2/3}\frac{h^2}{2m} = \varepsilon_F$$

$$\varepsilon_F = \left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{2/3}\frac{\hbar^2}{2m}$$

First order correction:

In Eq. 1 replace $(\ln z)^{-2} = \left(\frac{\varepsilon_F}{kT}\right)^{-2}$

and solve for $\mu = kT \ln z$ keeping the second term in the expansion.

$$\underbrace{\frac{N}{V}}_{n} = \frac{4\pi g}{3}\left(\frac{2m}{h^2}\right)^{3/2}\underbrace{\left(kT\ln z\right)^{3/2}}_{\mu^{3/2}}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2+\cdots\right]$$

$$1 \simeq \underbrace{\frac{4\pi g}{3n}\left(\frac{2m}{h^2}\right)^{3/2}}_{1/\varepsilon_F^{3/2}}\mu^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$$

Then

$$\mu = \left[\frac{\varepsilon_F^{3/2}}{1+\frac{\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2}\right]^{2/3} = \varepsilon_F\left[1-\frac{2}{3}\frac{\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2\right] =$$

$$= \varepsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$$

Using $\mu(T)$ we can obtain $U(T)$ and $P(T)$:

$$U = \frac{3}{2} NkT \frac{f_{5/2}(z)}{f_{3/2}(z)} \simeq \frac{3}{\cancel{2}} NkT \frac{\cancel{8}^{4}\ \cancel{3}\sqrt{\cancel{\pi}}}{\cancel{15}_{5}\ \cancel{\sqrt{\pi}}^{1}\ \cancel{4}} \ln z \times$$

from last time

$$\times \frac{\left(1 + 5\frac{\pi^2}{8}(\ln z)^{-2} + \cdots\right)}{\left(1 + \frac{\pi^2}{8}(\ln z)^{-2} + \cdots\right)} =$$

$\left(\frac{kT}{\varepsilon_F}\right)^2$ $\left(\frac{kT}{\varepsilon_F}\right)^2$

$$\simeq \frac{3}{5} N \underbrace{kT \ln z}_{\varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)} \left(1 + \frac{5\pi^2}{8}(\ln z)^{-2} + \cdots\right)\left(1 - \frac{\pi^2}{8}(\ln z)^{-2} + \cdots\right)$$

$$\simeq \frac{3}{5} N \varepsilon_F \left[1 + \frac{5}{12}\pi^2\left(\frac{kT}{\varepsilon_F}\right)^2\right] \quad (2)$$

$$P = \frac{2}{3}\frac{U}{V} = \frac{2}{5}\frac{N}{V}\varepsilon_F\left(1 + \frac{5\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots\right)$$

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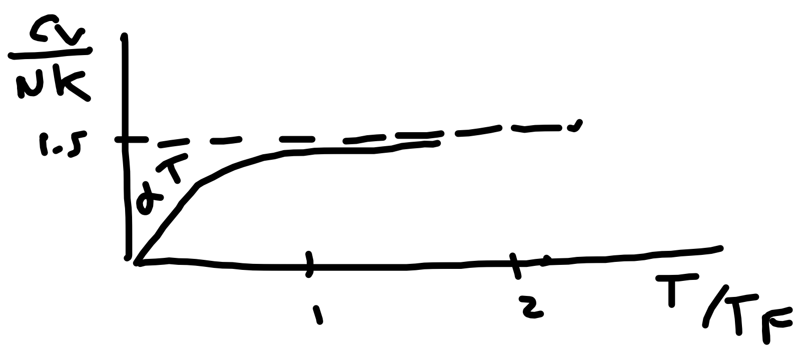
$$\frac{c_V}{Nk} = \frac{1}{Nk}\frac{\partial U}{\partial T} \overset{②}{=} \frac{3}{5}\frac{\varepsilon_F}{k}\frac{5}{12}\pi^2\frac{2k^2T}{\varepsilon_F^2} + \cdots =$$

$$= \frac{\pi^2 kT}{2\varepsilon_F} + \cdots \propto T$$ linear in T!



$T_F = \frac{\varepsilon_F}{k}$ Fermi temperature

$$\frac{F}{N} = \mu - \frac{PV}{N} = \frac{3}{5}\varepsilon_F\left[1 - \frac{5\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2 + \cdots\right]$$

$$S = \frac{U-F}{T} = \frac{\pi^2}{2}\frac{kT}{\varepsilon_F} + \cdots \qquad S \to 0 \text{ if } T \to 0.$$

Magnetic behavior of the ideal Fermi gas:

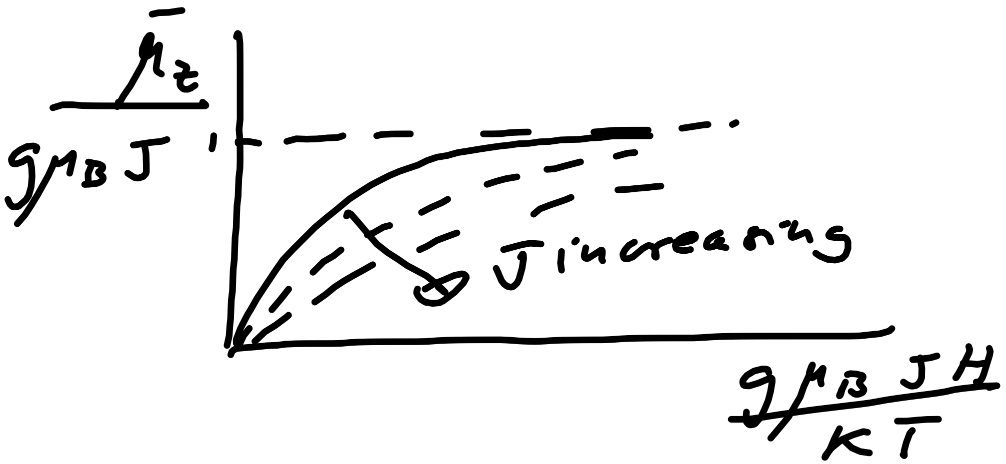
Consider fermions (with spin $\frac{1}{2}$) in a magnetic field.

We want to find $\bar{M}(\bar{B},T)$ and

$$\chi = \lim_{\bar{B}\to 0} \frac{\bar{M}}{\bar{B}}.$$

Before we studied this problem and

found



Then $\chi = \lim_{B \to 0} \frac{M}{B} \propto \frac{1}{T}$ Curie's law.

For $g = 2$ and $J = \frac{1}{2}$ $\chi_0 = \frac{n\mu^{*2}}{kT}$

This is the behavior to be expected in the non-degenerate regime $n\lambda^3 \ll 1$. However, in the degenerate regime only a fraction of the fermions can react to the B-field.

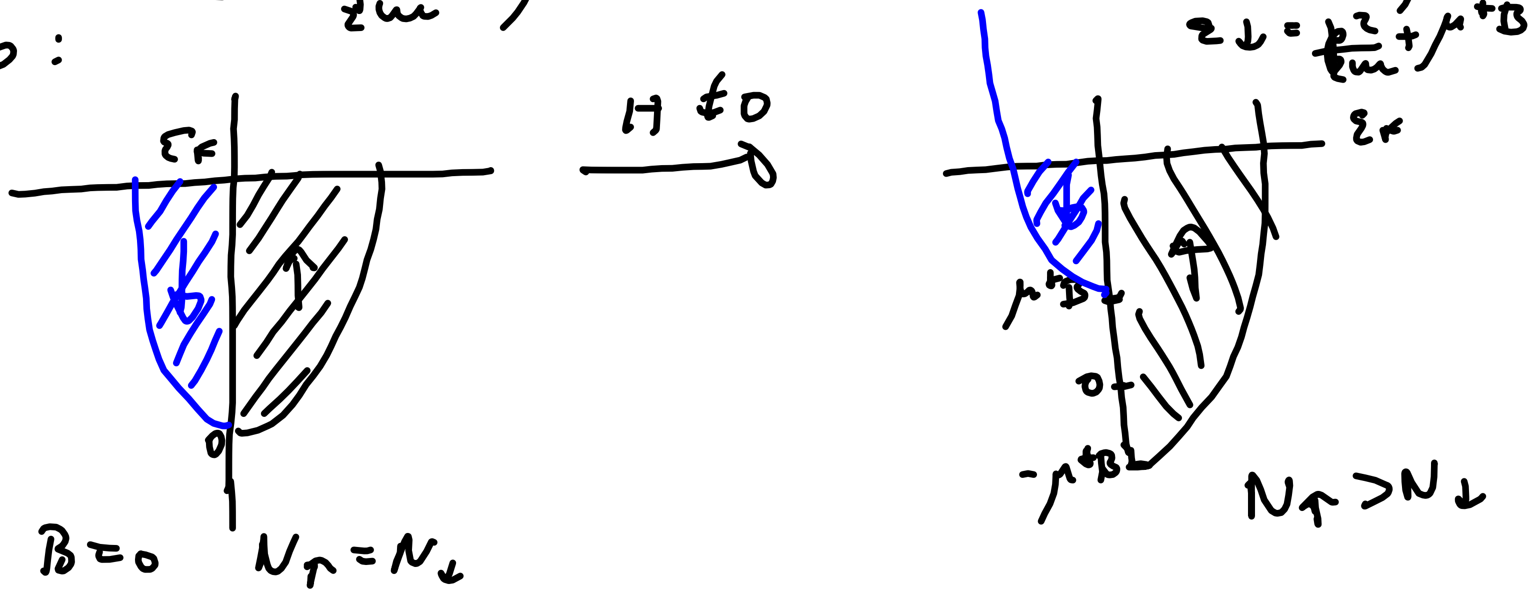
$$\varepsilon = \frac{p^2}{2m} - \bar{\mu}^* \cdot \bar{B}$$

if $s = 1/2$

$$\varepsilon_\uparrow = \frac{p^2}{2m} - \mu^* B$$

$$\varepsilon_\downarrow = \frac{p^2}{2m} + \mu^* B$$

$T = 0$:



Then (remember that $N\big|_{T=0} = g \frac{V 4\pi}{3h^3} p_F^3 =$

$p_F^3 \to (2m\varepsilon_F)^{3/2}$

g: 2 when ↑ and ↓ are together but $g=1$ for ↑ or ↓

$$N^+ = \frac{4\pi V}{3h^3}\left[2m\left(\varepsilon_F + \mu^* B\right)\right]^{3/2}$$

$$N^- = \frac{4\pi V}{3h^3}\left[2m\left(\varepsilon_F - \mu^* B\right)\right]^{3/2}$$

$$M = \mu^*(N^+ - N^-) = \frac{4\pi V}{3h^3}(2m)^{3/2}\left[\left(\varepsilon_F + \mu^* B\right)^{3/2} - \left(\varepsilon_F - \mu^* B\right)^{3/2}\right]$$

$$\chi_0 = \lim_{B\to 0}\left(\frac{M}{VB}\right) = \frac{4\pi\mu^{*2}(2m)^{3/2}\varepsilon_F^{1/2}}{h^3} = \text{constant.}$$

3

Since $\varepsilon_F = \left(\frac{6\pi^2 n}{g}\right)^{2/3}\frac{\hbar^2}{2m}$ and $g=2$

we can replace in 3

$$\chi_0 = \frac{3}{2}n\frac{\mu^{*2}}{\varepsilon_F} \quad \text{constant}$$

Notice that

$$\frac{\chi_0}{\chi_\infty} = \frac{\frac{3}{2}n\mu^{*2}/\varepsilon_F}{n\mu^{*2}/kT} = \frac{3}{2}\frac{kT}{\varepsilon_F} \approx O\left(\frac{kT}{\varepsilon_F}\right)$$

Now let's obtain χ for $T > 0$.

The energy of the system is given by:

$$E_n = \sum_p \left[\left(\frac{p^2}{2m} - \mu^* B \right) n_p^+ + \left(\frac{p^2}{2m} + \mu^* B \right) n_p^- \right] =$$

$$= \sum_p (n_p^+ + n_p^-) \frac{p^2}{2m} - \mu^* B (N^+ - N^-)$$

and

canonical

$$Z(N) = \sum_{\{n_p^+\},\{n_p^-\}}' e^{-\beta E_n}$$

constraints:

$n_p^+, n_p^- = 0$ or 1

$\sum_p n_p^+ + \sum_p n_p^- = N^+ + N^- = N$

Then we can write:

$N^- = N - N^+$

$$Z(N) = \sum_{N^+=0}^{N} \left[e^{\beta\mu + B(N^+ - N^-)} \left\{ \sum_{\{n_p^+\}}'' e^{-\beta \sum_p \frac{p^2}{2m} n_p^+} \right\} \right.$$

$\sum_p n_p^+ = N^+$

$$\left. \left\{ \sum_{\{n_p^-\}}''' e^{-\beta \sum_p \frac{p^2}{2m} n_p^-} \right\} \right] =$$

$\sum_p n_p^- = N^- = N - N^+$

$$= \sum_{N^+=0}^{N} \left[e^{\beta\mu + B(2N^+ - N)} \left\{ \sum_{\{n_p^+\}}'' e^{-\beta \sum_p \frac{p^2}{2m} n_p^+} \right\} \left\{ \sum_{\{n_p^-\}}''' e^{-\beta \sum_p \frac{p^2}{2m} n_p^-} \right\} \right]$$

Define

$$Z_0(N) = \sum_{\{n_p\}}{}' \; e^{-\beta \sum_p \frac{p^2}{2m} n_p} = e^{-\beta F_0(N)}$$

$\sum_p n_p = N$

represents Z for a system of ideal spinless fermions.

Then:

$$Z(N) = e^{-\beta \mu^* B N} \sum_{N^+=0}^{N} e^{2\beta\mu^* B N^+} Z_0(N^+) Z_0(N-N^+)$$

$$\therefore \ln Z(N) = -\beta \mu^* B N + \ln \sum_{N^+=0}^{N} e^{2\beta\mu^* B N^+} Z_0(N^+) Z_0(N-N^+)$$

$$\frac{\ln Z(N)}{N} = -\beta\mu^+ B + \frac{1}{N}\ln \underbrace{\sum_{N^+=0}^{N} e^{\beta[2\mu^+ B N^+ - F_0(N^+) - F_0(N-N^+)]}}_{\text{we will keep only the largest term with } N^+=\bar{N}^+ \text{ because all the other terms are much smaller.}}$$

To find $\bar{N}^+$ we solve:

$$\left.\cancel{\beta} 2\mu^+ B - \cancel{\beta}\frac{\partial F_0(N^+)}{\partial N^+} - \cancel{\beta}\frac{\partial F_0(N-N^+)}{\partial N^+}\right|_{N^+=\bar{N}^+} = 0 \qquad (4)$$

since $\frac{\partial F}{\partial N} = \mu$

Then (4) becomes:

$$2\mu^* B = \mu_0(\bar{N}^+) - \mu_0(N - \bar{N}^+) \qquad (5)$$

Define:

$$M = \mu^*(\bar{N}^+ - \bar{N}^-) = \mu^*(\bar{N}^+ - (N - \bar{N}^+)) =$$

$$= \mu^*(2\bar{N}^+ - N) = \mu^* N r \qquad (0 \leq r \leq 1)$$

Let's rewrite (5) as:

$$2\mu^* B = \mu_0\left(\frac{(1+r)}{2}N\right) - \mu_0\left(\frac{(1-r)}{2}N\right)$$

$$2\frac{(1+r)}{2}N - N$$
$$= N + rN - N =$$
$$= rN$$