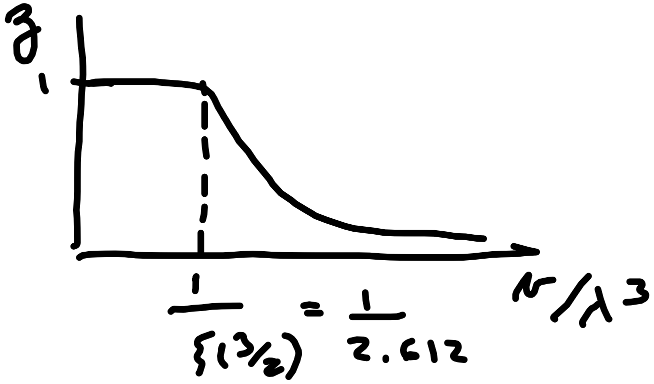
11/3

Last time:



$T \leq T_c \Rightarrow \frac{v}{\lambda^3} = \frac{1}{g_{(3/2)}(z)} < \frac{1}{2.612}$

For $\frac{v}{\lambda^3} >> 1$

$g_{3/2}(z) << 1 \Rightarrow z << 1$

$g_{3/2}(z) \sim z << 1$

Pressure in the Bose gas:

$P(T)\big|_{V/N}$

We found last time.:

$$P = \frac{kT}{\lambda^3} g_{5/2}(z) = \frac{kTN}{V} \frac{g_{5/2}(z)}{g_{3/2}(z)} \quad \text{①}$$

$$\frac{N}{V} = \frac{g_{3/2}(z)}{\lambda^3} \Rightarrow \lambda^3 = g_{3/2}(z) \frac{V}{N}$$

If $T < T_c \Rightarrow z \sim 1$

$$P(T) = \frac{kT}{\lambda^3}\, g_{5/2}(z=1) = \frac{kT}{h^3}(2\pi m kT)^{3/2}$$

$\propto T^{5/2}$ and independent of V.

$$\therefore \; K_T = -\frac{1}{V}\frac{\partial V}{\partial P}\Big|_T \longrightarrow \infty \quad \text{incompressible.}$$

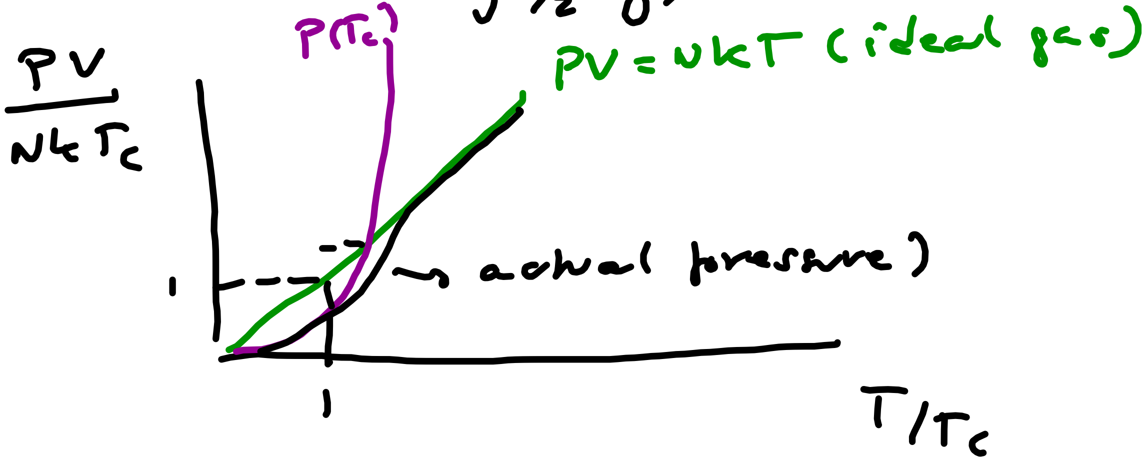
$$P(T_c) = \frac{kT_c}{\lambda^3}\,\zeta(5/2) \overset{\textcircled{1}}{=} \frac{\zeta(5/2)}{\zeta(3/2)}\,\frac{N}{V}\,kT_c \quad (\text{using that } z=1)$$

$$= 0.5134\,\frac{NkT_c}{V}$$ about $\frac{1}{2}$ P for ideal gas!

The Bose-Einstein condensate gas makes a lower P than an ideal gas.

For $T > T_c$

$$P \overset{①}{=} \frac{N}{V} \frac{kT g_{5/2}(z)}{g_{3/2}(z)}$$



we found last time that

$$P = \frac{2}{3} \frac{U}{V}$$

$$\frac{PV}{NkT_c} = \frac{2}{3} \frac{UV}{VNkT_c}$$

$$= \frac{2}{3} \frac{U}{NkT_c}$$

curve for $P \propto$ curve for $U(T)$.

Then $c_V(T)$ is $\left.\frac{\partial U}{\partial T}\right|_V$ and from the figure $c_V \to 0$ for $T \to 0$ (good!)

For $T << T_c$ $\lambda^3 = \frac{h^3}{(2\pi m kT)^{3/2}}$

$$P(T) = \frac{kT}{\lambda^3} \zeta(5/2) = \frac{(kT)^{5/2}}{h^3} (2\pi m)^{3/2} \zeta(5/2)$$

$$\therefore \ U = \frac{3}{2} PV = \frac{3}{2} \frac{(2\pi m)^{3/2}}{h^3} V (kT)^{5/2} \zeta(5/2)$$

$$c_V = \left.\frac{\partial U}{\partial T}\right|_V = \frac{15}{4} \zeta(5/2) \frac{(2\pi m)^{3/2}}{h^3} V k^{5/2} T^{3/2} \propto T^{3/2}$$

$C_V \propto T^{3/2}$ goes to 0 as $T \to 0$.

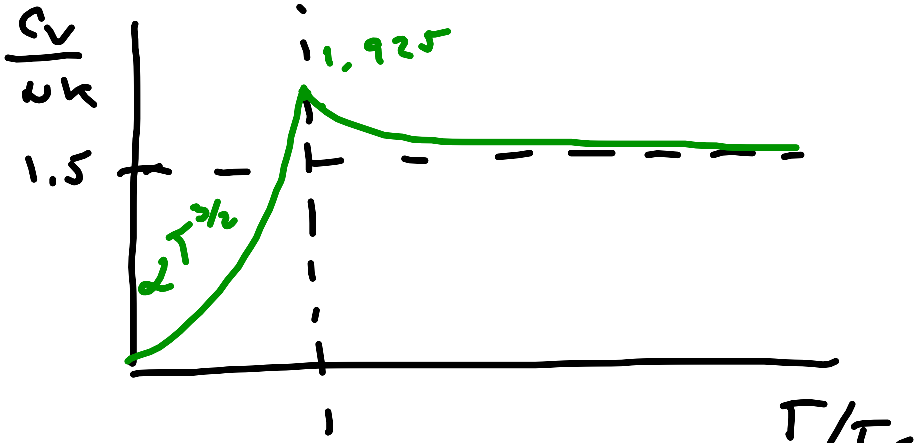
At $T = T_c$ using that $N = \frac{V(2\pi m k T)^{3/2}}{h^3}\zeta(3/2)$ from ②

$$\frac{C_V}{Nk} = \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)} = 1.925 > 1.5 = \frac{3}{2}$$

classical value.

Remember that for $T > T_c$ we found

$$\frac{C_V}{Nk} = \frac{3}{2}\left[\underbrace{1 + 0.0884\left(\frac{\lambda^3}{v}\right) + \cdots}_{>1}\right] \xrightarrow[T\to\infty]{} \frac{3}{2}$$

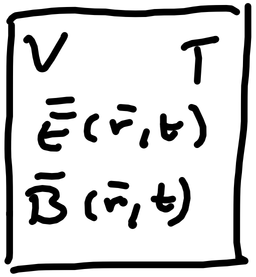


Actual "non-interacting" B. E. condensation was observed in 1995 at $T \sim 10^{-9}$ K.

called "λ" behavior.

Similar to c_v vs T for He^4 which pointed towards the fact that B. E. condensation occurs in He^4. However, the particles of He^4 are interacting and more is needed to explain superfluidity

Black body radiation:



$$\bar\nabla \cdot \bar A = 0 \qquad \bar E\,,\ \bar B\,,\ \bar A$$

$$\bar E = -\frac{\partial \bar A}{\partial t} \qquad \bar B = \bar\nabla \times \bar A$$

$$\nabla^2 E = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \nabla^2 B = \varepsilon \mu \frac{\partial^2 B}{\partial t^2}$$

$$\bar A(\bar r,t) = \sum_{\bar k,\mu=1,2} \Big(\hat e^{(\mu)}(\bar k)\, a^{(\mu)}_{\bar k}(t)\, e^{i\bar k\cdot\bar r} + \hat e^{*(\mu)}(\bar k)\, a^{*(\mu)}_{\bar k}(t)\, e^{-i\bar k\cdot\bar r} \Big)$$

$$W = \frac{1}{2}\int_V \left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) dV =$$

$$= \frac{\varepsilon V}{16}\sum_{k,\mu}\left(\dot{a}^2_{k,\mu} + \frac{k^2}{\mu_0\varepsilon_0} a^2_{k,\mu}\right) \equiv H$$

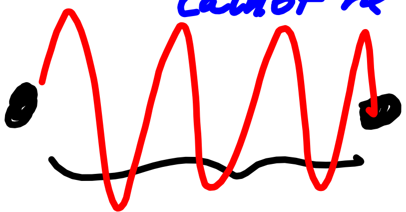
$\omega = c k$ c : speed of light

This was well known in the XIX'th century.

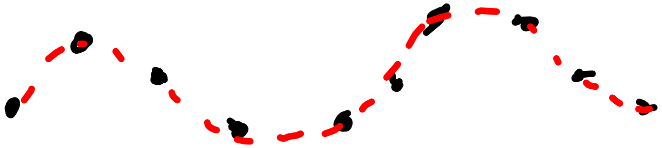
However, using classical mechanics to solve the problem led to the "ultraviolet catastrophe".

The problem was that there was no reason to cut-off the possible values of ω. $E = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$ it could become infinitely large.

Notice that in solids it was simple to find a cut off because:



This short wave length cannot be "followed" by the molecules



sound wave in solid

a : intermolecular distance in the solid (lattice constant.

Then the total energy was bounded in solids but in the black body Raijleigh and jeans obtained:

$$\int_0^\infty u(\omega)\,d\omega \longrightarrow \infty$$

ultraviolet catastrophe.

Planck got the idea that the energy accessible to the harmonic oscillators was quantized so that

$$\varepsilon_s = (n_s + \tfrac{1}{2}) \hbar \omega_s$$

(Planck did not know about the zero point energy).

$n_s = 0, 1, 2, \ldots$

ω_s identifies the oscillators

(M-B statistics).

Notice that classically:

$$Z_1^{\text{class.}}(V,T) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta\left(\frac{1}{2} m \omega_s^2 q^2 + \frac{1}{2m} p^2\right)} \frac{dq\, dp}{h} =$$

$$= \frac{1}{h}\left(\frac{2\pi}{\beta m \omega^2}\right)^{1/2}\left(\frac{2\pi m}{\beta}\right)^{1/2} = \frac{kT}{\hbar\omega_s}$$

$$\therefore \quad Z_N^{class}(\beta) = \left(\frac{kT}{\hbar\omega_s}\right)^N$$ if all N oscillators have $\omega_s = \omega$.

Using Planck's proposal:

$$Z_1^{q.m}(V,T) = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega_s} = \frac{e^{-\frac{1}{2}\beta\hbar\omega_s}}{1-e^{-\beta\hbar\omega_s}} =$$

$$= \frac{e^{-\frac{\hbar\omega_s}{2kT}}}{1-e^{-\hbar\omega_s/kT}}$$

and

$$Z_N^{g.m}(\beta) = \frac{e^{-\frac{N\hbar\omega_s}{kT}}}{\left(1 - e^{-\frac{\hbar\omega_s}{kT}}\right)^N}$$

this was the "novelty" introduced by Planck.

(Notice that if $\hbar\omega_s << kT$ $Z_N^{g.m} \to \frac{1}{\left(1-1+\frac{\hbar\omega_s}{kT}\right)^N}$

$$= \left(\frac{kT}{\hbar\omega_s}\right)^N = Z_N^{clas}$$

Then now

$$U_1 = -\frac{\partial \ln Z_1^{g.m}}{\partial\beta} = \underbrace{\frac{1}{2}\hbar\omega_s}_{\text{0-point energy}} + \underbrace{\frac{\hbar\omega_s e^{-\beta\hbar\omega_s}}{1 - e^{-\beta\hbar\omega_s}}}_{\text{B. E distribution.}} =$$

$$= \langle \varepsilon_s \rangle$$

Let's calculate the number of oscillators per unit volume in the cavity with frequency between ω and $\omega + d\omega$ (i.e. $g(\omega)\, d\omega$);

We know that:

# of polarizations

$$dn(k)\, d^3k = \overset{\text{\# of polarizations}}{2}\, \frac{V}{(2\pi)^3}\, d^3k =$$

# of oscillators with $\bar{k} \in (\bar{k}, \bar{k} + d\bar{k})$

$\hookrightarrow k_i = \frac{2\pi}{L_i} n_i \therefore \# \frac{k_i^3}{\frac{(2\pi)^3}{L^3}}$

$$= \frac{2V}{(2\pi)^3} k^2 dk\, \underbrace{4\pi}_{\text{no angular dependence}} = \frac{V k^2 dk}{\pi^2} =$$

$k = \omega / c$

$dk = d\omega / c$

$$= \frac{V}{\pi^2} \frac{\omega^2}{c^2} \frac{d\omega}{c} = \frac{V \omega^2}{\pi^2 c^3} d\omega$$

Then dividing by V we get

$$g(\omega)\,d\omega = \frac{\omega^2}{\pi^2 c^3}\,d\omega$$

Then the energy density associated to the frequency range $(\omega, \omega + d\omega)$ is

$$u(\omega)\,d\omega = \frac{\hbar\omega\;\omega^2\;d\omega}{\pi^2 c^3\left(e^{\beta\hbar\omega}-1\right)} \xrightarrow[\hbar\omega \ll kT]{} \frac{\hbar\omega^3\,d\omega}{\pi^2 c^3(\beta\hbar\omega)}$$

$$\propto \omega^2\,d\omega$$

leads to classical result at high T.

$\int_0^\infty u(\omega)\,d\omega$ is finite!

Bose's approach: he used q.m. and worked directly with "photons" the particles associated to the "quanta" of e.m. radiation. He distributed the photons over the available energy levels focusing on the probability that level ε_s is occupied by n_s photons at a time. He used M-B. statistics with q.m. oscillators and obtained:

$$\langle n_s \rangle = \frac{\sum_{n_s=0}^{\infty} n_s \, e^{-n_s \frac{\hbar \omega_s}{kT}}}{\sum_{n_s=0}^{\infty} e^{-n_s \frac{\hbar \omega_s}{kT}}} = \frac{1}{e^{\frac{\hbar \omega_s}{kT}} - 1}$$

$$\langle \varepsilon_s \rangle = \hbar \omega_s \langle n_s \rangle = \frac{\hbar \omega_s}{e^{\hbar \omega_s / kT} - 1}$$

identical to Planck's.

Einstein's approach: He considered the statistics of the photons and the energy levels together. He considered the bosons indistinguishable (implicit Bose).

He used the same treatment that we did using Lagrange multipliers (we used α and β).

But because N is not fixed for photons Einstein had $\alpha = 0$ (it means $\mu = 0$) and then he obtained

$$\langle n_\varepsilon \rangle = \frac{1}{e^{\beta\varepsilon} - 1} = \frac{1}{e^{\hbar\omega/kT} - 1}.$$

1 oscillator in eigenstate n_s with $\varepsilon_s = (n_s + \frac{1}{2})\hbar\omega_s$ is equivalent to energy state $\varepsilon_s = \hbar\omega_s$ occupied by n_s photons and it is equivalent to the average energy $\langle \varepsilon_s \rangle$ of an oscillator corresponds to the mean occupation $\langle n_s \rangle$ of the corresponding energy level.