9/22

Last time:

N magnetic dipoles in a H-field

canonical - classical:

$$Z_N(\beta) = \left[\frac{4\pi}{\beta\mu H}\sinh\beta\mu H\right]^N$$

$$\bar{\mu}_z = \frac{M_z}{N} = \langle\mu\cos\theta\rangle = \frac{kT}{N}\frac{\partial \ln Z_N}{\partial H} = -\frac{1}{N}\frac{\partial F}{\partial H}\Big|_T$$

Let's calculate

$$\bar{\mu}_z = \frac{kT}{N}\frac{\partial \ln Z^N}{\partial H} = \frac{kT}{N}\frac{\partial \ln\left(\frac{4\pi \sinh \beta\mu H}{\beta\mu H}\right)^N}{\partial H}\Bigg|_\beta =$$

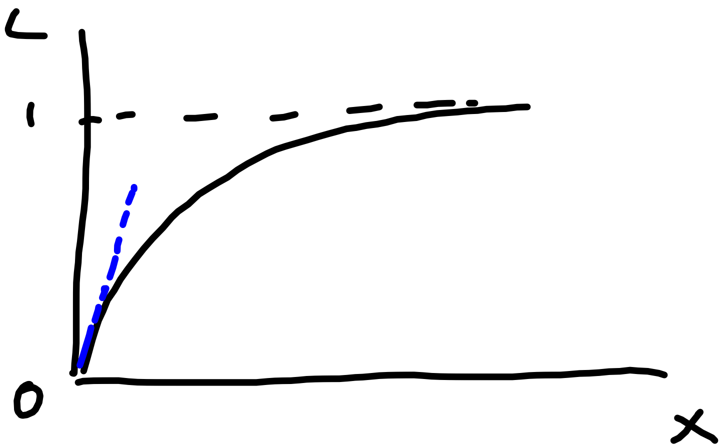
$$= \frac{kT}{\cancel{N}}\cancel{N}\frac{\cancel{4\pi}\left(\beta\mu\frac{\cosh\beta\mu H}{\cancel{\beta\mu H}} - \frac{\sinh\beta\mu H}{\cancel{\beta\mu} H^{\cancel{2}}}\right)}{\frac{\cancel{4\pi}\sinh\beta\mu H}{\cancel{\beta\mu H}}} =$$

$$= \cancel{kT}\cancel{\beta}\mu\left(\coth\beta\mu H - \frac{1}{\beta\mu H}\right) =$$

$$= \mu\left(\coth\beta\mu H - \frac{1}{\beta\mu H}\right) = \mu L(\beta\mu H)$$

$\hookrightarrow$ Langevin function.

$$L(x) = \coth x - \frac{1}{x}$$



If $N_0 = \frac{N}{V}$ dipole density

$M_{z_0} = N_0 \bar{\mu}_z = N_0 \mu L(x)$ $\quad x = \beta \mu H = \frac{\mu H}{kT}$

magnetization per unit volume.

For $x \gg 1 \Rightarrow \mu H \gg kT$

low temperature

$M_{z_0} = N_0 \mu$ saturation all dipoles are allign with H.

For $x \ll 1 \Rightarrow \mu H \ll kT$ high temperature limit.

$L(x) \simeq \frac{x}{3} - \frac{x^3}{45} + \cdots$

$M_{z_0} = N_0 \mu \left(\frac{x}{3} - \frac{x^3}{45}\right) \simeq \frac{N_0 \mu^2 H}{3kT} \propto H$

The magnetic susceptibility is

$$\chi_T = \lim_{H\to 0} \frac{\partial M_{z0}}{\partial H}\Big|_T \simeq \frac{N_0 \mu^2}{3kT} = \frac{C}{T}$$

Curie's law

Same problem using quantum mechanics:

$$\bar{\mu} = \underbrace{g \frac{e}{2mc}}_{\text{gyromagnetic ratio}} \bar{\ell}$$

Lande's factor

angular momentum

$$\ell^2 = J(J+1)\hbar^2$$

$J = \frac{1}{2}, \frac{3}{2}, \ldots$

or $J = 0, 1, 2, \ldots$

$$g = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

for spins $S = J$, $L = 0$, $g = 2$

for orbital motion $S = 0$, $L = J$, $g = 1$

$$\bar{\mu} = g \frac{e}{2mc} \bar{e}$$

then

$$\mu^2 = g^2 \frac{e^2}{2^2 m^2 c^2} J(J+1) \hbar^2 = g^2 \mu_B^2 J(J+1)$$

with $\mu_B = \frac{e\hbar}{2mc}$ Bohr's magneton.

then

$$\mu_z = g \mu_B m$$

$m = -J, -J-1, \ldots J-1, J$

$2J+1$ possible orientations

$$Z_1(\beta) = \sum_{m=-J}^{J} e^{\beta g \mu_B m H} = \sum_{m=-J}^{J} e^{\frac{mx}{J}} =$$

$x = \beta g \mu_B J H$

geometric sum

$$= e^{-x} \frac{\left\{ e^{(2J+1)\frac{x}{J}} - 1 \right\}}{e^{x/J} - 1} =$$

$$= \frac{\sinh\left\{ \left(1 + \frac{1}{2J}\right) x \right\}}{\sinh\left(\frac{1}{2J} x\right)}$$

$$M_z = N \bar{\mu}_z = \frac{N}{\beta} \frac{\partial \ln Z_1(\beta)}{\partial H} =$$

as in the classical case.

$x = \beta g \mu_B J H$

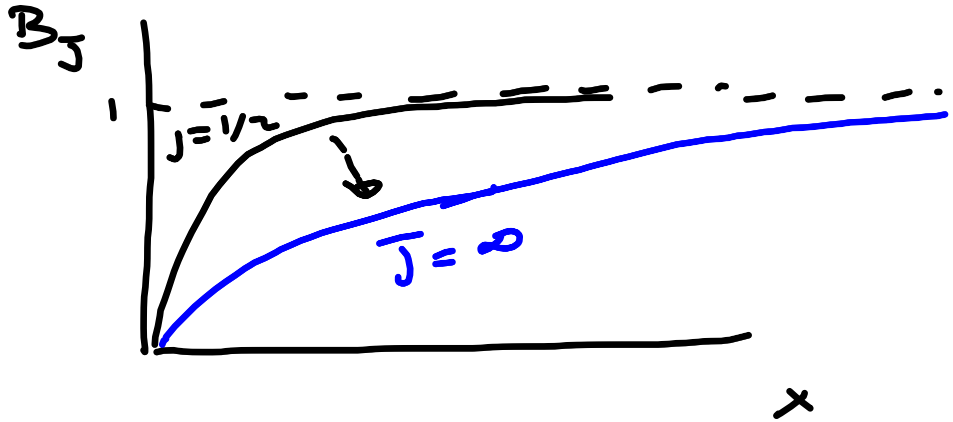
$$= N(g\mu_B J)\left[\left(1+\frac{1}{2J}\right)\coth\left(1+\frac{1}{2J}\right)x - \right.$$

$$\left. - \frac{1}{2J}\coth\left(\frac{1}{2J}x\right)\right]$$

then

$$\bar{\mu}_z = g\mu_B J \underbrace{B_J(x)}_{\text{Brillouin's function.}}$$

$$B_J(x) = \left(1+\frac{1}{2J}\right)\coth\left[\left(1+\frac{1}{2J}\right)x\right] - \frac{1}{2J}\coth\frac{x}{2J}$$



as J increases it takes larger values of x to reach saturation.

$$B_J(x) \simeq \frac{1}{3}\left(1+\frac{1}{J}\right)x$$

Then

$\bar{\mu}_z = g\mu_B J$ saturated when $x >> 1$

(compare with $\bar{\mu}_z = \mu$ for classical).

For $x \ll 1$:

$$\bar{\mu}_z \approx \frac{g\mu_B J}{3}\left(1+\frac{1}{J}\right)\overbrace{\beta g \mu_B J H}^{x} =$$

$$= \frac{g^2 \mu_B^2 J(J+1) H}{3kT}$$

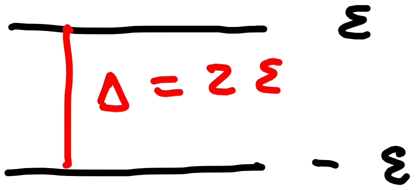
Then

$$\frac{\chi_T}{N} = \lim_{H\to 0}\left(\frac{\partial \bar{\mu}_z}{\partial H}\right)_T \propto \frac{1}{T}$$ Curie's law.

$$\chi \propto \frac{1}{T}$$ with $C_J = \frac{N_0 \mu^2}{3k}$ $\quad \mu^2 = g^2 \mu_B^2 J(J+1)$

Negative temperatures in magnetic systems:

Consider a system of dipoles with $J = \frac{1}{2}$.



This is a two level system

$$\varepsilon = \mu_B H$$

N dipoles.

$$Z_N(\beta) = \left(e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right)^N = \left(2\cosh\beta\varepsilon\right)^N$$

$$F = -kT \ln Z_N = -kTN \ln\left(2\cosh\frac{\varepsilon}{kT}\right)$$

$$S = -\frac{\partial F}{\partial T}\Big|_{H,N} = Nk\left[\ln\left(2\cosh\frac{\varepsilon}{kT}\right) - \right.$$

$$\left. - \frac{\varepsilon}{kT}\tanh\frac{\varepsilon}{kT}\right]$$

if $kT \ll \varepsilon$ (we expect $S=0$) see that

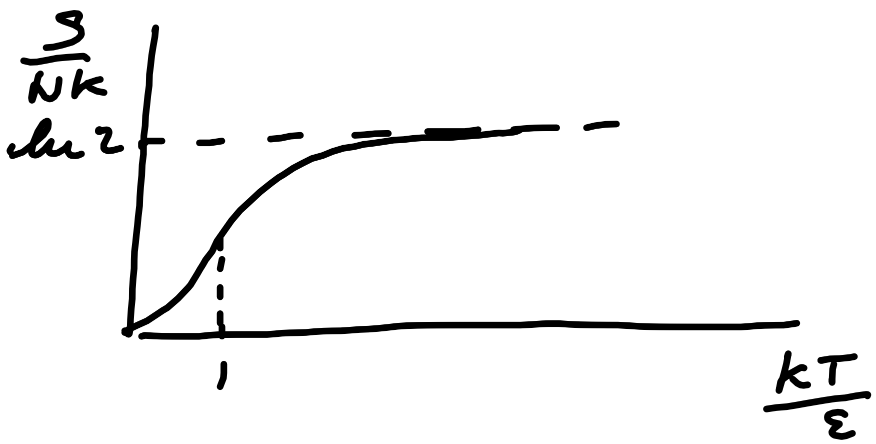
$$S = Nk\left[\ln \cancel{2}\left(\frac{e^{\varepsilon/kT} + \cancel{e^{-\varepsilon/kT}}}{\cancel{2}}\right)\right] -$$

$$- \frac{\varepsilon}{kT}\frac{\cancel{e^{\varepsilon/kT}}}{\cancel{e^{\varepsilon/kT}}} = Nk\left[\frac{\varepsilon}{kT} - \frac{\varepsilon}{kT}\right] = 0$$

If $kT >> \varepsilon$ we expect $S = kN \ln 2$

because there are 2^N accessible states.

We see that $S = Nk \ln 2$ (from our expression).



$$U = F + TS = -N\varepsilon \tanh \frac{\varepsilon}{kT}$$

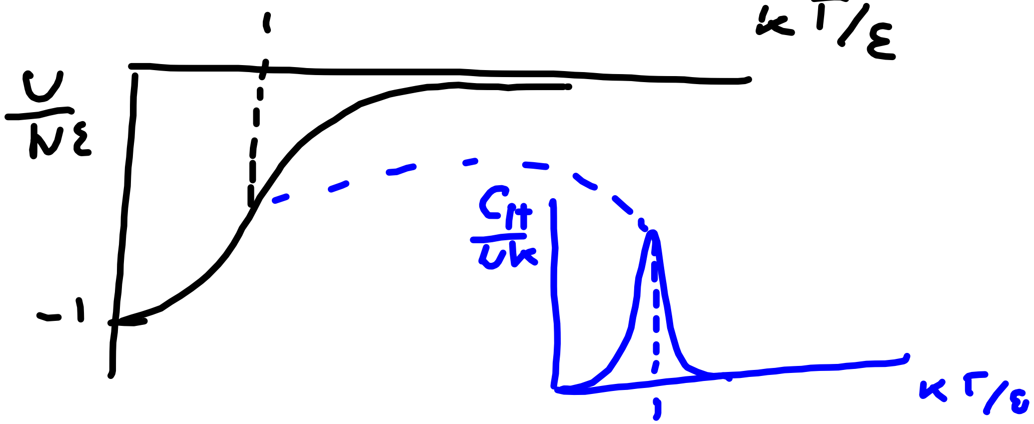
$$\lim_{kT \ll \varepsilon} U = -N\varepsilon$$

$$\lim_{kT \gg \varepsilon} U = 0$$

$$M = -\frac{\partial F}{\partial H}\bigg|_T = N\mu_B \tanh \frac{\varepsilon}{kT}$$

$M = N\mu_B$ for $\varepsilon \gg kT$

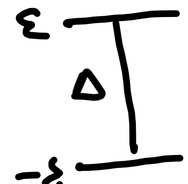
$M = 0$ for $\varepsilon \ll kT$



$$C_H = \frac{\partial U}{\partial T}\bigg|_H = Nk\left(\frac{\varepsilon}{kT}\right)^2 \operatorname{sech}^2 \frac{\varepsilon}{kT} \quad (1)$$

Let's rewrite ① in terms of $\Delta = 2\varepsilon$

Replace $\varepsilon = \frac{\Delta}{2}$ in ①:



$$C_H = Nk\left(\frac{\varepsilon}{kT}\right)^2 \operatorname{sech}^2\frac{\varepsilon}{kT} = Nk\left(\frac{\Delta}{2kT}\right)^2 \frac{4}{\left(e^{\beta\varepsilon}+e^{-\beta\varepsilon}\right)^2}$$

$$= Nk\left(\frac{\Delta}{kT}\right)^2 \frac{1}{e^{-2\beta\varepsilon}\left(e^{2\beta\varepsilon}+1\right)^2} =$$

$$= Nk\left(\frac{\Delta}{kT}\right)^2 e^{\frac{\Delta}{kT}}\left(1+e^{\Delta/kT}\right)^2$$

→ Schottky anomaly

it leads to a peak in C vs T that appears in systems with a Δ between g.s. and first excited state.

Negative temperature?

For systems such like a gas E always increases with T then $T > 0$ since otherwise $e^{-\beta E}$ would diverge in Z.

However, if a system has an upper bound for E, like in a system of dipoles, then $e^{-\beta E}$ will be finite even if $T < 0$.

ε ———————— N_2

$-\varepsilon$ ———————— N_1

$$N_1 + N_2 = N$$

$$U = N_2 \varepsilon - N_1 \varepsilon$$

$$\therefore N_2 - N_1 = U/\varepsilon$$

$$S(U,N) = k \frac{N!}{N_1!\,(N-N_1)!} = k \ln \frac{N!}{\left(\frac{N+U/\varepsilon}{2}\right)!\left(\frac{N-U/\varepsilon}{2}\right)!}$$

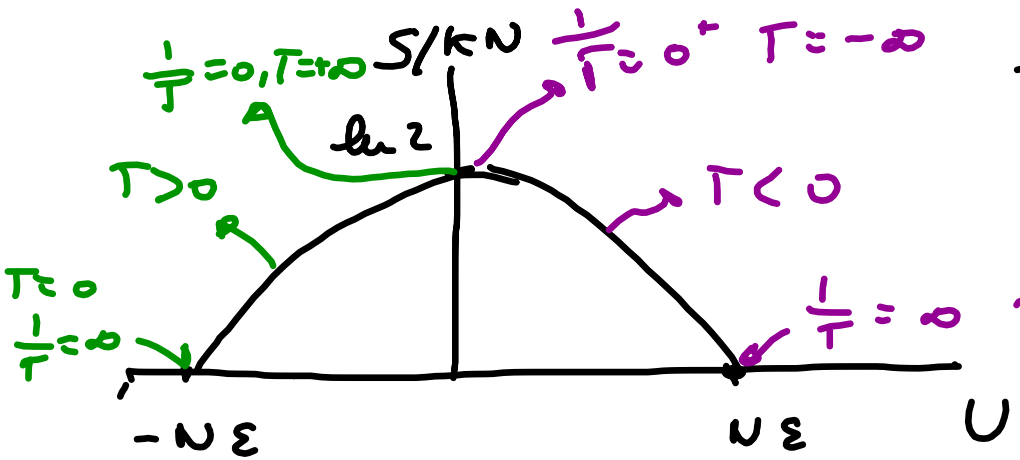
$$\approx k \left\{ N \left[\ln N - \frac{1}{2} \ln \left(\frac{N+U/\varepsilon}{2}\right) - \frac{1}{2} \ln \left(\frac{N-U/\varepsilon}{2}\right) \right] - \frac{U}{2\varepsilon} \ln \left(\frac{N+U/\varepsilon}{2}\right) + \frac{U}{2\varepsilon} \ln \left(\frac{N-U/\varepsilon}{2}\right) \right\}$$

Stirling

You can check that If $U = -N\varepsilon \Rightarrow$

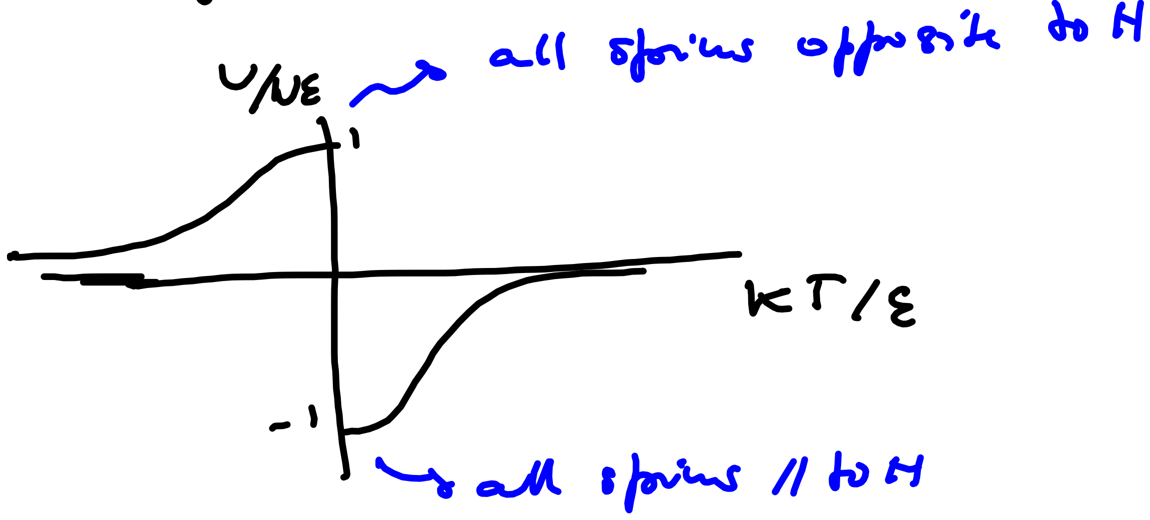
$$S = k \ln\left(\frac{N!}{N_1!\,0_2!}\right) = k \ln 1 = 0$$

If $U = 0 \Rightarrow N_1 = N_2 = \frac{N}{2}$ and replacing in the expression you'll get $S = kN \ln 2$



$$\frac{1}{T} = \frac{\partial S}{\partial E} \equiv \frac{\partial S}{\partial U}$$

Now let's plot $U/N\varepsilon$ vs kT/ε allowing T to be negative:



The state with $U = N\varepsilon$ can be achieved by a rapid reversal of H field and it last for a few minutes. Then the system has negative T.

After a while the system thermalizes with the lattice loosing energy. Thus, its temperature comes down. This shows that negative temperatures are "hotter" than positive ones.