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Ideal Bose systems:

non-interacting bosonic particles (integer spin).

when $n\lambda^3 = \frac{n h^3}{(2\pi m kT)^{3/2}} \gtrsim 1$ this quantum behavior occurs.

$n\lambda^3$ will be an expansion parameter because

$n\lambda^3 \to 0 \Rightarrow$ classical.

Notice that $\frac{n}{T^{3/2}}$ determines the quantum effects

since $n\lambda^3 \gtrsim 1$ if n is large (high density) and/or T is low (also small mass m increases quantum effects)

Thermodynamics of the Bose Gas:

$$\frac{PV}{kT} = q = \ln \mathcal{Z} \overset{a=-1}{=} -\sum_{\varepsilon_s} \ln\left(1 - z e^{-\beta \varepsilon_s}\right) \quad (1)$$

and

$$N = \sum_{\varepsilon_s} \langle n_{\varepsilon_s} \rangle \overset{a=-1}{=} \sum_{\varepsilon_s} \frac{1}{z^{-1} e^{\beta \varepsilon_s} - 1} \quad (2)$$

Clearly $z e^{-\beta \varepsilon} < 1$

Since $\varepsilon_s = \varepsilon_s(k)$ and $\Delta k \ll 1$ we will replace

$\sum \longrightarrow \int$:

We found previously (2.4.7)

$$a(\varepsilon)\,d\varepsilon = \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}\,d\varepsilon$$

Notice that $a(\varepsilon=0)=0$ this is ok for the ideal gas but now at very low T $\varepsilon=0$ may be important.

Also in quantum mechanics each state has to have weight equal to 1.

Then we are going to consider the contributions of the state with $\varepsilon=0$ separately:

Let's go from $\sum \longrightarrow \int$ in ①:

$$\frac{P\cancel{V}}{kT} = -\frac{2\pi\cancel{V}}{h^3}(2m)^{3/2}\int_0^\infty \varepsilon^{1/2}\ln\left(1 - z e^{-\beta\varepsilon}\right)d\varepsilon \qquad (1)$$

$$-\frac{1}{V}\ln(1-z)$$

for $\varepsilon = 0$

doing the same in ②:

$$\frac{N}{V} = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}\,d\varepsilon}{z^{-1}e^{\beta\varepsilon}-1} + \frac{1}{V}\,\frac{z}{1-z} \qquad (2)$$

Notice that for $z = e^{\mu/kT} << 1$ (classical limit) the term with $\varepsilon = 0$ contribution is negligible.

But if $z \to 1$ $(\Rightarrow \mu \to 0) \Rightarrow \dfrac{z}{1-z} = N_0 =$ # of particles with $\varepsilon = 0$

Now N_0 could become very large as $T \to 0$.

This is called Bose-Einstein condensation.

However, since $N_0 = \dfrac{z}{1-z} \Rightarrow$

$$N_0(1-z) = z$$
$$N_0 - N_0 z = z$$
$$N_0 = z(1+N_0)$$
$$\therefore z = \frac{N_0}{1+N_0}$$

$$\therefore \frac{\ln(1-z)}{V} = \frac{\ln\left(1 - \frac{N_0}{1+N_0}\right)}{V} = \frac{\ln(1/(1+N_0))}{V} =$$
$$= -\frac{\ln(1+N_0)}{V} \lesssim \frac{\ln N}{V}$$

this means that we can drop the $\frac{\ln(1-z)}{V}$ term.

Let's define $x = \beta\varepsilon$ then ① becomes:

$$\frac{P}{kT} = -2\pi\frac{(2mkT)^{3/2}}{h^3}\int_0^\infty \underbrace{x^{1/2}}_{u'}\,\underbrace{\ln(1-ze^{-x})}_{v}\,dx =$$

$$= -2\pi\frac{(2mkT)^{3/2}}{h^3}\left[\underbrace{\frac{2}{3}x^{3/2}\ln(1-ze^{-x})\Big|_0^\infty}_{0} - \int_0^\infty \frac{2}{3}\frac{x^{3/2}ze^{-x}dx}{1-ze^{-x}}\right]$$

$$= \underbrace{\frac{4}{3\sqrt{\pi}}}_{\Gamma(5/2)^{-1}}\frac{1}{\lambda^3}\int_0^\infty \frac{x^{3/2}dx}{z^{-1}e^x-1} = \frac{1}{\lambda^3}g_{5/2}(z)$$

and in ②:

$$\frac{N - N_0}{V} = \frac{\sqrt{\pi}}{\sqrt{\pi}} \, \frac{2\pi\,(2mkT)^{3/2}}{h^3} \int_0^\infty \frac{x^{1/2}\,dx}{z^{-1}e^x - 1} \equiv \frac{1}{\lambda^3}\, g_{3/2}(z)$$

$\Gamma(3/2)^{-1}$ λ^{-3}

with $\lambda = \dfrac{h}{(2\pi mkT)^{1/2}}$

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}\,dx}{z^{-1}e^x - 1} = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \cdots$$

$\Gamma(n) = (n-1)!$

$\Gamma(\tfrac{1}{2}+n) = \dfrac{(2n-1)!!}{2^n}\sqrt{\pi}$ $\quad\therefore\quad$ $\Gamma(3/2) = \dfrac{\sqrt{\pi}}{2}$

$\Gamma(5/2) = \dfrac{3}{4}\sqrt{\pi}$

Eliminating z from ① and ② will provide the equation of state.

$$U = -\frac{\partial \ln \mathcal{Z}}{\partial \beta}\bigg|_{z,V} = kT^2 \frac{\partial}{\partial T}\left(\frac{PV}{kT}\right)_{z,V} =$$

$$= kT^2 V g_{5/2}(z) \underbrace{\frac{d\lambda^{-3}}{dT}}_{\lambda^{-3}\frac{3}{2}T^{-1}} = \frac{3}{2} kT \underbrace{\frac{V}{\lambda^3} g_{5/2}(z)}_{P/kT},$$

$$\therefore \quad U = \frac{3}{2} PV \quad \text{or} \quad \boxed{P = \frac{2}{3}\frac{U}{V}} \quad ③$$

The equation of state will be given by a Virial expansion in powers of $n\lambda^3 = \frac{\lambda^3}{v}$:

If z is small

(1) $$\frac{P}{kT} \simeq \frac{1}{\lambda^3}\left(z + \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \cdots\right)$$

(2) $$\frac{N-N_0}{V} \underset{z \ll 1}{\approx} \frac{N}{V} \approx \frac{1}{\lambda^3}\left(z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots\right)$$

$$\therefore \frac{\lambda^3}{v} = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots$$

Now we can invert the series writing z in powers of λ^3/v :

$$z = c_1 \frac{\lambda^3}{v} + c_2 \left(\frac{\lambda^3}{v}\right)^2 + \cdots = \sum_{\ell=1}^{\infty} c_\ell \left(\frac{\lambda^3}{v}\right)^\ell \quad (4)$$

Now Replacing (4) in (1) and dividing by n we obtain:

$$\frac{P}{nkT} = \sum_{\ell=1}^{\infty} \underbrace{a_\ell}_{\text{virial coefficients}} \left(\frac{\lambda^3}{v}\right)^{\ell-1}$$

Virial expression for the equation of state - (See HW#11).

Notice that $a_1 = 1$ and $a_2 = -\frac{1}{4\sqrt{2}}$ (HW)

$$\therefore \frac{PV}{NkT} \cong 1 - \frac{1}{4\sqrt{2}} \frac{\lambda^3}{v} \quad \text{eq. of state.}$$

Notice that

$$\frac{C_V}{Nk_B} = \frac{1}{Nk_B}\frac{\partial U}{\partial T}\Big|_{V,N} \overset{\text{since } U=\frac{3}{2}PV}{=} \frac{3}{2}\frac{\partial}{\partial T}\left(\frac{PV}{Nk_B}\right)_{N,V} =$$

$$= \frac{3}{2}\frac{\partial}{\partial T}\left[T\sum_{\ell=1}^{\infty} a_\ell \left(\frac{\lambda^3}{v}\right)^{\ell-1}\right]_{N,V} = \qquad k \equiv k_B$$

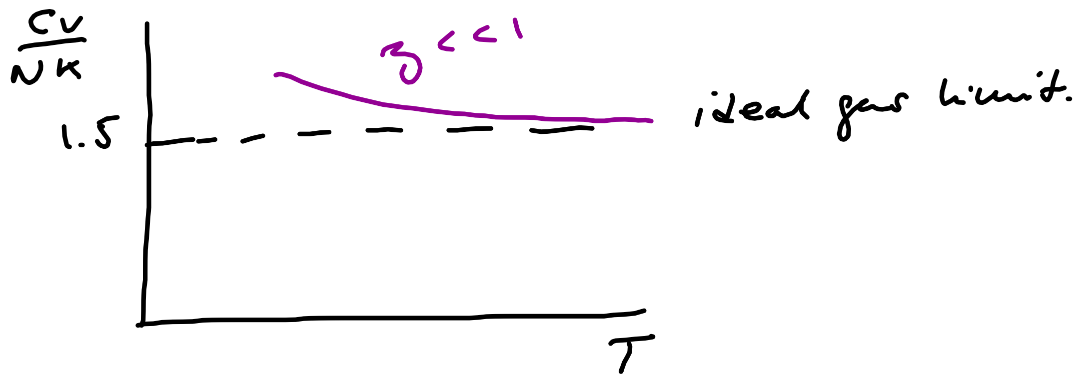
$$= \frac{3}{2}\frac{\partial}{\partial T}\left\{\sum_{\ell=1}^{\infty}\frac{a_\ell}{v^{\ell-1}}\left[\frac{h^3}{(2\pi m k)^{3/2}}\right]^{\ell-1}\underbrace{\frac{T}{(T^{3/2})^{\ell-1}}}\right\} =$$

$$= \frac{3}{2}\sum_{\ell=1}^{\infty}\frac{a_\ell}{v^{\ell-1}}\left[\frac{h^3}{(2\pi m k)^{3/2}}\right]^{\ell-1}\left(\frac{5}{2}-\frac{3\ell}{2}\right)T^{3/2-\frac{3}{2}\ell} \quad \Big\{ T^{1-\frac{3}{2}\ell+\frac{3}{2}} = T^{\frac{5}{2}-\frac{3\ell}{2}}$$

$$= \frac{3}{2}\sum_{\ell=1}^{\infty} a_\ell\left(\frac{\lambda^3}{v}\right)^{\ell-1}\frac{(5-3\ell)}{2} = \frac{3}{2}\left(1+\frac{1}{8\sqrt{2}}\frac{\lambda^3}{v}+\cdots\right)$$

(5)

Then:



For $T \to \infty \Rightarrow \lambda^3 = 0 \quad \therefore \frac{C_V}{NK} = \frac{3}{2}$ (classical)

We will see that for $T \to 0$ $C_V \to 0$ that means that at some point C_V will have to come down.

Notice that expression ⑤ for c_V is not very useful to find c_V for $T \to 0$ because we have assumed $N >> N_0$ which is not the case for $T \to 0$. Then, we will use a different approach:

We found that

N_e : # of particles not in the ground state.

$$\frac{\underbrace{N - N_0}_{N_e}}{V} = \frac{1}{\lambda^3} g_{3/2}(z)$$

$$N_e = \frac{V}{\lambda^3} g_{3/2}(z) = V \frac{(2\pi m k T)^{3/2}}{h^3} g_{3/2}(z)$$

Notice that

$$g_\nu(z) = z + \frac{z^2}{2^\nu} + \dots \quad \therefore \quad g_{3/2}(z) \simeq z + \frac{z^2}{2^{3/2}} + \dots$$

$$z = e^{\mu/kT} \qquad \mu < 0 \qquad \therefore \quad \underset{\text{high } T}{0} \le z \le \underset{\text{low } T}{1}$$

If $z \ll 1$

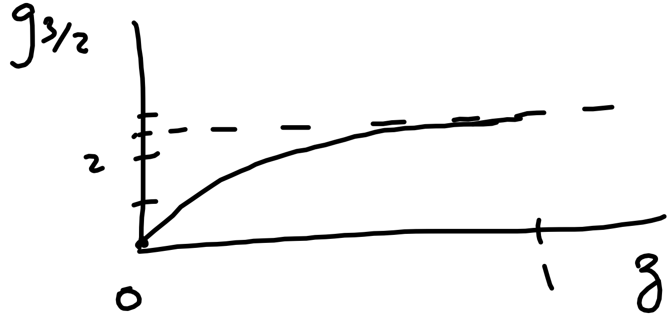
$$N_e = (N - N_0) = \frac{V}{\lambda^3} g_{3/2}(z) = N$$ (because T is high)

and we found that

$$N = V \frac{2\pi}{h^3} (2m)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{z^{-1} e^{\beta\varepsilon} - 1} + \underbrace{\overbrace{\frac{1}{V} \frac{z}{1-z}}^{\text{vanishes for } z \ll 1}}_{N_0}$$

As z goes from 0 to 1:

$$g_{3/2}(z=1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots \simeq \zeta(3/2) \simeq 2.612$$



We see that it has to satisfy that:

$$N_e = \frac{V}{\lambda^3} g_{3/2}(z) \leq \frac{V}{\lambda^3} \zeta(3/2) = f(T, V)$$

If $N < N_e$ allowed by T and V there is no problem. (A)

But if T and V are such that $N > N_e$ what happens? (B)

In situation (A) most of the N particles are distributed among the allowed excited states and $N_0 << 1$ and z is determined from asking that

$N_e \simeq N$.

In situation B if $N > N_e$ we will get that

$$N_e \equiv \frac{V}{\lambda^3} \zeta(3/2)$$

and the rest of the particles will be in $\varepsilon = 0$

then

$$N_0 = N - N_e = N - \frac{V}{\lambda^3} \zeta(3/2)$$

and now we find z using that

$$N_0 = \frac{z}{1-z} \Rightarrow (1-z) N_0 = z \Rightarrow N_0 = z(1+N_0)$$

if $\frac{1}{N_0} \ll 1$

$$\therefore z = \frac{N_0}{1+N_0} = \frac{N_0}{N_0(1+\frac{1}{N_0})} \approx 1\left(1 - \frac{1}{N_0}\right) \approx 1$$

When this happens we get Bose-Einstein condensation.

Notice that B. E. cond. occurs when

$$N > \frac{V}{\lambda^3} \zeta(3/2) = \frac{V\, T^{3/2}}{h^3} (2\pi m k)^{3/2} \zeta(3/2)$$

If we keep N and V constant we can find

T_c for the system.