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Bose Einstein condensation (cont.)

Last time we found that B.E. cond. occurs if

$$N > N_e \equiv \frac{V}{\lambda^3}\underbrace{\zeta(3/2)}_{2.61} = \frac{V}{h^3} T^{3/2} (2\pi m k)^{3/2} \zeta(3/2)$$

when $N = N_e \Rightarrow T = T_c$ $\therefore$

$$T_c = \left[\frac{N}{V}\frac{h^3}{(2\pi m k)^{3/2}\zeta(3/2)}\right]^{2/3} = \left(\frac{N}{V}\right)^{2/3}\frac{h^2}{2\pi m k\,\zeta(3/2)^{2/3}}$$

If $T < T_c \Rightarrow N_0 \neq 0$ and B.E. cond. occurs.

$$T_c = T_c\left(\frac{N}{V}, m\right)$$

For $T < T_c$ the system is in a mixture of 2 phases: i) Normal phase with

$$N_e = N\left(\frac{T}{T_c}\right)^{3/2}$$

D/
$$N = V\, T_c^{3/2} \underbrace{\left(\frac{2\pi m k}{h^2}\right)^{3/2}}_{1/\lambda^3 / T^{3/2}} \zeta(3/2)$$

and
$$N_e = V\, T^{3/2} \left(\frac{2\pi m k}{h^2}\right)^{3/2} \zeta(3/2) = N\left(\frac{T}{T_c}\right)^{3/2} \checkmark$$

ii) Condensate phase that has

$N_0 = N - N_e$ particles with $\varepsilon = 0$.

$$\frac{N_0}{N} = \frac{N - N_e}{N} = 1 - \frac{N_e}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

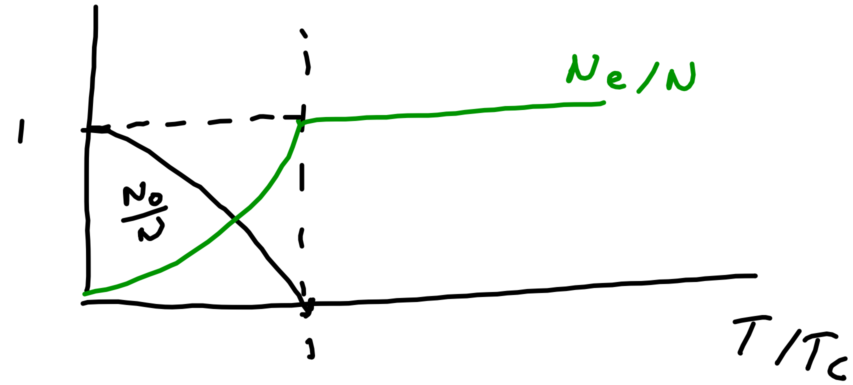
if $T \to T_c$:

$$\frac{N_0}{N} \approx 1 - \left(\frac{T + T_c - T_c}{T_c}\right)^{3/2} = 1 - \left(\frac{T_c + \Delta}{T_c}\right)^{3/2} =$$

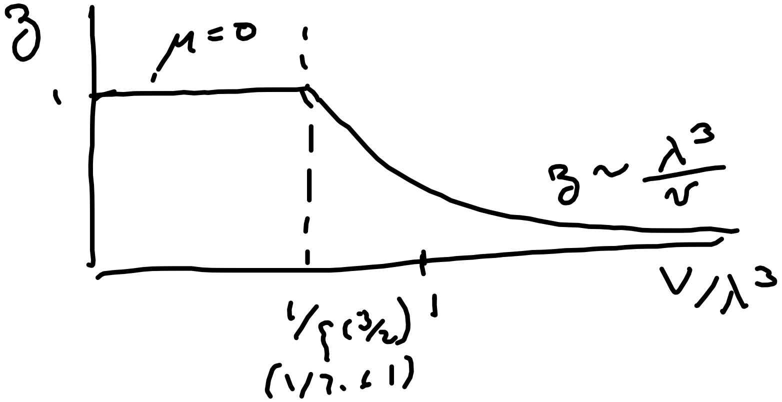
$\Delta \to T - T_c$

$$= 1 - \left(1 + \frac{\Delta}{T_c}\right)^{3/2} \approx 1 - \left(1 + \frac{3}{2}\frac{\Delta}{T_c}\right) \approx -\frac{3}{2}\frac{\Delta}{T_c} =$$

$$= -\frac{3}{2}\frac{T - T_c}{T_c} = \frac{3}{2}\frac{(T_c - T)}{T_c}$$



Also: $z = e^{\mu/kT}$



$T > T_c$ we found that

$$n = \frac{N_e}{V} = \frac{1}{\lambda^3} g_{3/2}(z) \simeq$$

$$\simeq \frac{1}{\lambda^3} (z + \cdots)$$

$$\frac{N_e}{V} = \frac{z}{\lambda^3} \quad \text{for } z \text{ small}$$

$$z = \frac{\lambda^3}{v}$$

For $T \to 0$ $\quad N_e \to 0 \Rightarrow \frac{V}{\lambda^3} = 0$ because $\zeta(3/2) > 0$

$\left(N_e = \frac{V}{\lambda^3} \zeta(3/2) \right)$

$T \to T_c \quad N_e \to N \Rightarrow \frac{V}{\lambda^3} = \frac{N_e}{\zeta(3/2)}$

$$\therefore \quad 0 \leq \frac{V}{\lambda^3} \leq \frac{1}{\zeta(3/2)}$$

$$0 \leq T \leq T_c .$$

Behavior of $P(T)$ for a Bose gas:

For $T < T_c \Rightarrow z \approx 1 \quad \therefore \quad P(T) = \frac{kT}{\lambda^3} g_{5/2}(z) \approx \frac{kT}{\lambda^3} \zeta_{5/2} \propto T^{5/2}$

$\therefore \kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P}\Big|_T \to \infty$ incompressible

compressibility

$P = P(T)$ independent of V.

$\hookrightarrow \propto 1/T^{3/2}$

$$P(T_c) = \frac{kT_c}{\lambda^3}\zeta(5/2) = \left(\frac{2\pi m}{h^2}\right)^{3/2}(kT_c)^{5/2}\zeta(5/2) =$$

$$= \frac{\zeta(5/2)}{\zeta(3/2)}\frac{N}{V}kT_c = \underline{\underline{0.513}}4\,\frac{N}{V}kT_c$$

$$T_c = \frac{h^2}{2\pi m k}\left(\frac{N}{V\zeta(3/2)}\right)^{2/3}$$

$P = \frac{N}{V}kT$ for ideal gas

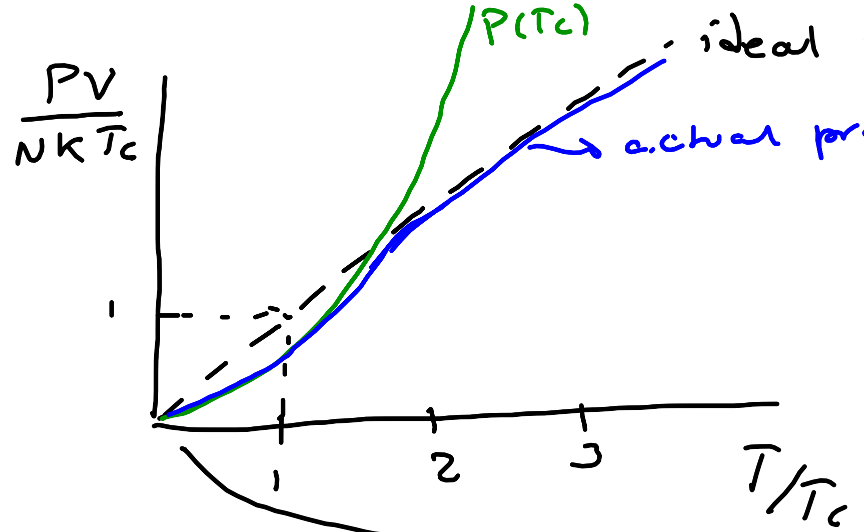
$P_{B\,gas\ at\ T_c} \sim \frac{1}{2}P_{ideal\ gas\ at\ T_c}$

For $T > T_c$:

$$P = \frac{N}{V}kT\,\frac{g_{5/2}(z)}{g_{3/2}(z)}$$

also remember that

$$g_{3/2}(z) = \frac{\lambda^3}{v}$$



→ actual pressure of the Bose gas.

$$P(T) = \frac{kT}{\lambda^3}\,\zeta(5/2)$$

Notice that: $\left.\frac{\partial U}{\partial T}\right|_V \to 0$ for $T \to 0$

$$P = \frac{2}{3}\frac{U}{V} \quad \text{(already found)}$$

$$\therefore \frac{PV}{NkT_c} = \frac{2U}{3NkT_c}$$

Since both curves for P and U are proportional to each other $C_V = \left.\frac{\partial U}{\partial T}\right|_V$ can be obtained from the figure.

For $T << T_c$ $\qquad \lambda^3 = \dfrac{h^3}{(2\pi m k T)^{3/2}}$

$$P(T) = \frac{kT}{\lambda^3} \zeta\left(\frac{5}{2}\right) = \frac{(kT)^{5/2}}{h^3} (2\pi m)^{3/2} \zeta(5/2)$$

$$\therefore \quad U = \frac{3}{2} PV = \frac{3}{2} \frac{(2\pi m)^{3/2}}{h^3} V (kT)^{5/2} \zeta(5/2)$$

$$\therefore \quad C_V = \frac{\partial U}{\partial T}\bigg|_V = \overbrace{\frac{3}{2} \times \frac{5}{2}}^{15/4} \frac{(2\pi m)^{3/2}}{h^3} V k^{5/2} T^{3/2} \zeta(5/2)$$

$$\propto T^{3/2} \qquad C_V(T \to 0) = 0$$

At $T = T_c$ since $N = V \dfrac{(2\pi m k T)^{3/2}}{h^3} \zeta(3/2)$

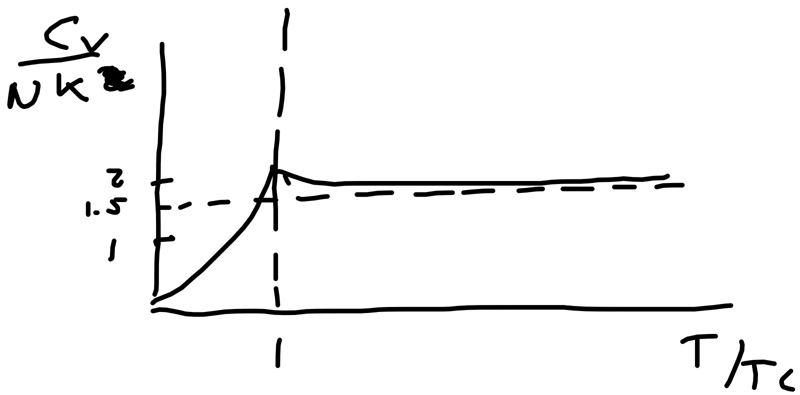
$$\frac{C_V}{Nk} = \frac{15}{4} \frac{\zeta(5/2)}{\zeta(3/2)} \simeq 1.925 > 1.5 \qquad (7.1.20)$$

$c_v(T_c) > \frac{3}{2}$ (ideal gas value).

For $T > T_c$ we already found that

$$\frac{c_v}{Nk} = \frac{3}{2}\left[1 + 0.884\frac{\lambda^3}{v} + \cdots\right] > \frac{3}{2}$$

but should go to $\frac{3}{2}$ for $T \to \infty$.



Similar to c_v vs T for He^4 which has BE condensation for $T_c = 3.13$ K (but the molecules interact, not ideal Bose gas).

He^4 has superfluidity due to interactions

First demonstration of <u>ideal</u> Bose gas was achieved in 1995 using "cold" atoms $T_c \sim 10^{-9}\,K$.

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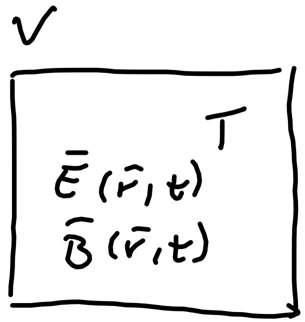
Blackbody radiation



$$\bar{\nabla}\cdot\bar{A} = 0$$

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} \qquad \bar{B} = \bar{\nabla}\times\bar{A}$$

$$\nabla^2 \bar{E} = \mu\varepsilon \frac{\partial^2 \bar{E}}{\partial t^2} \qquad \nabla^2 \bar{B} = \varepsilon\mu \frac{\partial^2 \bar{B}}{\partial t^2}$$

$$\bar{A}(\bar{r},t) = \sum_{\bar{k},\,\mu=\pm 1} \hat{e}^{(\mu)}(\bar{k})\, a_{\bar{k}}^{(\mu)}(t)\, e^{i\bar{k}\cdot\bar{r}} + \hat{e}^{(-\mu)}(\bar{k})\, a_{k}^{(-\mu)}(t)\, e^{-i\bar{k}\cdot\bar{r}}$$

$$\underset{\text{energy}}{W} = \frac{1}{2}\int_V \left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) dV =$$

$$= \frac{\varepsilon V}{16} \sum_{k,\mu} \underbrace{\left(\dot{a}^2_{\bar{k},\mu} + \frac{k^2}{\mu_0\varepsilon_0} a^2_{k,\mu}\right)}_{\text{looks like the energy for a harmonic oscillator}} \equiv H$$

$\omega = c k$ for photons

c: speed of light.

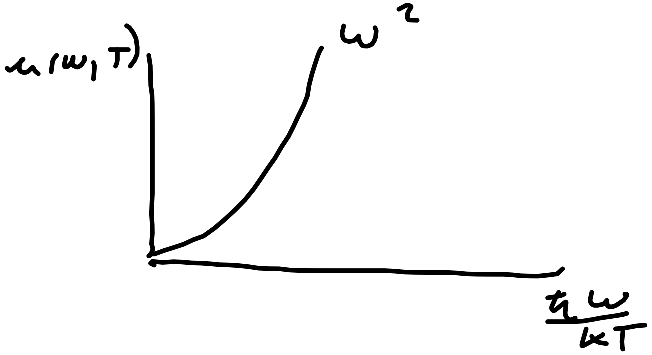
All this was known in the XIX century.

also $k = \frac{2\pi}{\lambda}$ λ : wavelength. $\omega = ck$

$\omega = \frac{c\, 2\pi}{\lambda}$

The problem was that any value of λ inside the cavity was allowed even very small values corresponding to ω very large.

Also notice that the density of energy for the oscillators goes like ω^2 :.



$$\int_0^\infty u(\omega,T)\, d\omega \longrightarrow \infty$$

ultraviolet catastrophe.

Notice that in solids (phonons):

The solution to the photon problem was given by Plank. He proposed that at low T the harmonic oscillator had discrete energies:

$$\varepsilon_s = (n_s + \tfrac{1}{2})\hbar\omega_s$$

$\frac{1}{2}$ was found later (not by Planck)

Since s distinguishes the oscillators M-B statistics will be used: In phase space:

$$Z_1^{class}(V,T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta\left(\frac{1}{2}m\omega_s^2 q^2 + \frac{1}{2m}p^2\right)} \frac{dq\,dp}{h} =$$

$$= \frac{1}{h}\left(\frac{2\pi}{\beta m\omega^2}\right)^{1/2}\left(\frac{2\pi m}{\beta}\right)^{1/2} = \frac{kT}{\hbar\omega_s}$$

(1D oscillator).

$$\therefore\ Z_N^{class}(\beta) = \left(\frac{kT}{\hbar\omega_s}\right)^N \quad \therefore\ \langle U\rangle = kTN$$

equipartition of energy.

Using Planck's prescription:

$$Z_1^{q.m}(V,T) = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega_s} = \frac{e^{-\frac{1}{2}\beta\hbar\omega_s}}{1-e^{-\beta\hbar\omega_s}} =$$

$$= \frac{e^{-\frac{\hbar\omega_s}{2kT}}}{1-e^{-\frac{\hbar\omega_s}{kT}}}$$

and

$$Z_N^{q.m}(\beta) = \frac{e^{-\frac{N\hbar\omega_s}{2kT}}}{\left(1-e^{-\frac{\hbar\omega_s}{kT}}\right)^N} \xrightarrow[kT \gg \hbar\omega_s]{} \left(\frac{kT}{\hbar\omega_s}\right)^N = Z_N^{class.}$$

Then

$$U_1 = -\frac{\partial \ln Z_1}{\partial \beta} = \frac{1}{2}\hbar\omega_s + \frac{\hbar\omega_s e^{-\beta\hbar\omega_s}}{1 - e^{-\beta\hbar\omega_s}} =$$

$$= \underbrace{\frac{1}{2}\hbar\omega_s}_{\text{0 point energy}} + \frac{\hbar\omega_s}{e^{\beta\hbar\omega_s} - 1} = \langle \varepsilon_s \rangle$$

→ B. E. distribution.

Next time we will find $g(\omega)d\omega$ # of oscillators per unit volume with frequency in $(\omega, \omega+d\omega)$ in order to obtain $u(\omega)$ and perform the integral.