11/30

- Last class: Tuesday December 5.
  - Finish Ch. 7
  - Give partial grades: HW + Midterms
  - Take home part of the final will be given.
  - In class part of the final: Thursday 12/14 @ 12^{30} PM in this room.

- Optional (not recommended) early in class final exam:
  - Monday December 4 @ 145 PM in room B07 Nielsen. Send me email if you will do this.
  - Come on Tuesday 12/5 to get at home part.

Last time: Black body radiation.

$$U_1 = -\frac{\partial \ln Z_1}{\partial \beta} = \frac{1}{2}\hbar\omega_s + \frac{\hbar\omega_s}{e^{\beta\hbar\omega_s}-1}$$

We want to find out $g(\omega)\,d\omega$: # of oscillators with $\omega \in (\omega, \omega+d\omega)$

$k_i = \frac{2\pi n_i}{L_i} \Rightarrow \Delta n_i = \frac{L_i}{2\pi} dk_i$

$$dn(k) = 2\, dn_x\, dn_y\, dn_z = 2\frac{V}{(2\pi)^3} d^3k =$$

2 polarization

no angular dependence

$$= \frac{2V4\pi k^2 dk}{(2\pi)^3} = \frac{V k^2 dk}{\pi^2}$$

$$= \frac{V}{\pi^2}\frac{\omega^2}{c^2}\frac{d\omega}{c} = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

$k = \omega/c$

$dk = \frac{d\omega}{c}$

Per unit volume:

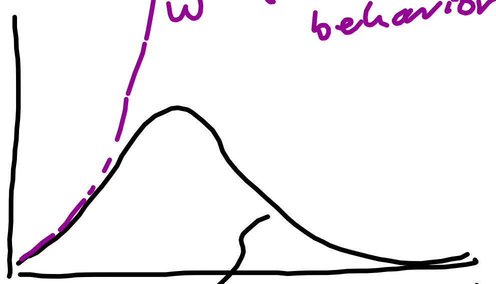
$$g(\omega)\,d\omega = \frac{dn(\omega)}{V} = \frac{\omega^2}{\pi^2 c^3}\,d\omega$$

Then the energy density can be obtained:

$$u(\omega)\,d\omega = \frac{\hbar\omega}{(e^{\beta\hbar\omega}-1)}\,g(\omega)\,d\omega = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{d\omega}{(e^{\beta\hbar\omega}-1)}$$

density of energy for photons with $\omega \in (\omega, \omega+d\omega)$

$$\hbar\omega \ll kT \;\Rightarrow\; \frac{\hbar\omega^3\,d\omega}{\pi^2 c^3(\cancel{1}+\beta\hbar\omega\cancel{-1})} \propto \omega^2$$



Now $\int_0^\infty u(\omega)\,d\omega$ is finite!

Bose's approach:

he used quantum mechanics and worked with photons. He distributed the photons over the available energy levels focusing on the probability that the level ε_s is occupied by n_s photons at a time. He used M-B statistics with q.m. distinguishable oscillators. He obtained that:

$$\langle n_s \rangle = \frac{\sum_{n_s=0}^{\infty} n_s \, e^{-n_s \frac{\hbar \omega_s}{kT}}}{\sum_{n_s=0}^{\infty} e^{-n_s \frac{\hbar \omega_s}{kT}}} = \frac{1}{e^{\hbar \omega_s / kT} - 1}$$

Notice that this is B.E. with $\mu = 0$.

=

and

$$\langle \varepsilon_s \rangle = \hbar \omega_s \langle n_s \rangle = \frac{\hbar \omega_s}{e^{\hbar \omega_s / kT} - 1}$$

identical to Planck's result (0 point energy missing).

He then obtained

$$g(\omega) d\omega = \frac{V \omega^2 d\omega}{\pi^2 c^3}$$

as we did before (same as Planck) by calculating the volume in phase space.

Einstein's approach:

he considered the quantum statistics of the photons and the energy levels together. He assumed that the photons were undistinguishable (he considered them "bosons").

He used the treatment for bosons that we used in class with the grand-canonical partition function. He used Lagrange multipliers (as we did) and obtained $\alpha = 0$. $\alpha \propto \mu \therefore \mu = 0$ because there is no constrain on N for photons.

Then he obtained:

$$\langle n_\varepsilon \rangle = \frac{1}{e^{\beta \varepsilon} - 1} = \frac{1}{e^{\frac{\hbar \omega}{kT}} - 1}$$

same as Bose and Planck.

- 1 oscillator in eigenstate n_s with energy $\varepsilon = (n_s + \frac{1}{2}) \hbar \omega_s = \varepsilon_s$ is equivalent to (Einstein)
- energy state $\varepsilon_s = \hbar \omega_s$ occupied by n_s photons. is equivalent to (Bose)
- the average energy $\langle \varepsilon_s \rangle$ of an "oscillator" corresponds to the mean occupation number $\langle n_i \rangle$ of the corresponding energy level. (Planck)

We know that $\mu = 0$ for photons $\Rightarrow$ $z = 1$.

Define $x = \frac{\hbar\omega}{kT} \Rightarrow dx = \frac{\hbar}{kT} d\omega$

$$u(\omega)\,d\omega = \frac{\hbar\,\omega^3\,d\omega}{(e^{\beta\hbar\omega}-1)\pi^2 c^3} = \frac{x^3 (kT)^3}{\hbar^3\,\pi^2 c^3} \cancel{\hbar} \frac{kT\,dx}{\cancel{\hbar}(e^x-1)} =$$

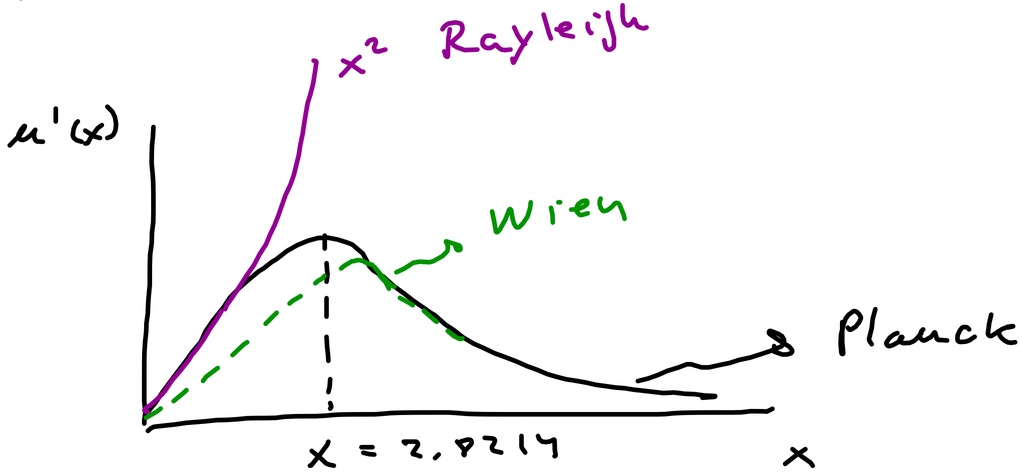
$$= \frac{x^3 (kT)^4}{\hbar^3 \pi^2 c^3} \frac{dx}{e^x-1} = \frac{(kT)^4}{c^3 \pi^2 \hbar^3} u'(x)\,dx$$

$$u'(x)\,dx = \frac{x^3}{e^x-1} dx$$

$$\lim_{x\to 0} u'(x) \simeq \frac{x^3}{\cancel{1}+x-\cancel{1}} = x^2$$

Rayleigh-Jones behavior (classical)

$\lim_{x \to \infty} u'(x) = x^3 e^{-x}$ Wien's law.



$\int_0^\infty u'(x)\,dx = \frac{\pi^4}{15} \approx 6.49$ Planck

$\int_0^\infty x^3 e^{-x}\,dx = 6$ Wien

$\int_0^\infty x^2\,dx \to \infty$ catastrophe!

Then

$$\frac{U}{V} = \int_0^\infty u(\omega)\, d\omega = \frac{(kT)^4}{\pi^2 c^3 \hbar^3} \underbrace{\int_0^\infty \frac{x^3\, dx}{e^x - 1}}_{\pi^4/15} = \frac{\pi^2 k^4 T^4}{15 \hbar^3 c^3} \qquad (1)$$

$\propto T^4$ Stephan-Boltzmann's law.

The radiation coming out from a small hole in the black-body cavity is found in the same way in which effusion of an ideal gas was calculated. We just need to replace $\langle n \rangle$ with u (density of particles replaced by density of energy).

$$R = \frac{1}{4} n \langle u \rangle = \frac{1}{4}\frac{U}{V} c = \frac{\pi^2 k^4 T^4}{60 \hbar^3 c^2} = \sigma T^4$$

replaced by U/V

of speed

replace by c

Stephan's constant

Thermodynamics (gas of photons with $\varepsilon = cp$ instead of $\varepsilon = \frac{1}{2m} p^2$ as for the classical ideal gas)

$$\mathcal{Z}(V, T, \mu = 0)$$

$$\ln \mathcal{Z}(V,T) = \frac{PV}{kT} = -\sum_{\varepsilon} \ln\left(1 - e^{-\varepsilon/kT}\right)$$

$$\xrightarrow{\int} -\int a(\varepsilon) \ln\left(1 - e^{-\varepsilon/kT}\right) d\varepsilon =$$

density of state

$$a(\varepsilon) = 2\,\frac{V 4\pi p^2 dp}{h^3} = \frac{2 V 4\pi \varepsilon^2/c^2\, d\varepsilon}{c h^3} = \frac{8\pi V \varepsilon^2 d\varepsilon}{c^3 h^3}$$

2 polarizations

$$\varepsilon = \hbar\omega = \underbrace{\hbar k}_{p} c = cp$$

$\neq$ dependence on ε for $a(\varepsilon)$

$$\ln \mathcal{Z}(V,T) = -\int_0^\infty \frac{8\pi V}{c^3 h^3} \underbrace{\varepsilon^2}_{u'} \underbrace{\ln\left(1-e^{-\varepsilon/kT}\right)}_{v} d\varepsilon =$$

$$= -\frac{8\pi V}{c^3 h^3}\left[\underbrace{\frac{1}{3}\varepsilon^3 \ln\left(1-e^{-\varepsilon/kT}\right)\Big|_0^\infty}_{0} - \frac{1}{3kT}\int \frac{\varepsilon^3 e^{-\varepsilon/kT}}{1-e^{-\varepsilon/kT}} d\varepsilon\right]$$

$$= \frac{8\pi V}{3c^3h^3kT}\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\varepsilon/kT}-1} = \frac{PV}{kT}$$

$x = \varepsilon/kT$

$$\therefore\ PV = \frac{8\pi V (kT)^{\cancel{3}\,4}}{3c^3h^3 \cancel{kT}} \underbrace{\int_0^\infty \frac{x^3 dx}{e^x-1}}_{\pi^4/15} = \frac{8\pi^5 V (kT)^4}{45 c^3 h^3} =$$

$$= \frac{1}{3}\left[\frac{\pi^2 V (kT)^4}{15 c^3 \hbar^3}\right] \overset{(1)}{=} \frac{U}{3}$$

$$F = E - TS = \cancel{TS} - PV + \underset{\overset{\shortparallel}{0}}{\cancel{\mu N}} - \cancel{TS} = -PV$$

$$= -\frac{U}{3}$$

$$\therefore \quad S = \frac{U - F}{T} = \frac{U + U/3}{T} = \frac{4}{3}\frac{\overset{\propto T^4}{U}}{T} \propto VT^3$$

$$\therefore \quad c_V = T\left(\frac{\partial S}{\partial T}\right)\Big|_V = T \overset{\text{constant}}{K} \frac{\partial (VT^3)}{\partial T}\Big|_V = TK\,3VT^2 =$$

$$= 3\underbrace{KVT^3}_{S} = 3S$$

$\therefore$ VT^3 constant $\Rightarrow$ $\Delta S = 0$ adiabatic.

Since $P \propto \frac{U}{V} \propto T^4 \quad \therefore \quad T \propto P^{1/4}$

$\therefore \; VT^3 \propto V P^{3/4}$ or $V^{4/3} P = \text{constant} \Rightarrow \Delta S = 0.$

What is $\bar{N}$ for the photon gas?

$$\bar{N} = \int a(\varepsilon)\, n(\varepsilon)\, d\varepsilon = \frac{V}{\pi^2 c^3} \underbrace{\int_0^\infty \frac{\omega^2\, d\omega}{e^{\hbar\omega/kT} - 1}}_{x = \hbar\omega/kT} =$$

$$= \frac{V\, 2\, \zeta(3) (kT)^3}{\pi^2 \hbar^3 c^3} \propto VT^3$$

Eq. 14b

Cosmic microwave background has $T = 2.7\,K$ is the black body remnant radiation from the big bang.

Phonons or atomic vibrations in crystals:

Consider a solid with N atoms located at $(x_1, x_2, \ldots, x_N)$. The minimum energy ϕ occurs ~~at~~ when the atoms are at $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$ (equilibrium position). Let $(x_i - \bar{x}_i) = u_i$ (atomic displacements). Then

Kinetic:

$$K = \frac{1}{2} m \sum_{i=1}^{3N} \dot{x}_i^2 = \frac{1}{2} m \sum_{i=1}^{3N} \dot{u}_i^2$$

Potential:

$$\phi \equiv \phi(x_i) = \phi(\bar{x}_i) + \sum_i \overbrace{\left.\frac{\partial \phi}{\partial x_i}\right|_{x_i = \bar{x}_i}}^{0} (x_i - \bar{x}_i) +$$

$$+ \frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{x_i = \bar{x}_i} (x_i - \bar{x}_i)(x_j - \bar{x}_j) + \cdots$$