SOLUTION:

Problem 1:

a) 
\[ F^b_a = g^{bc} F^c_b = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \]

\[ F^a_b = g^{ac} F^c_a = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \]

b) 
\[ F^a_b F^b_a = \sum_{a=0}^{3} \sum_{b=0}^{3} F^a_b F^b_a = 2(B^2 - E^2). \]

As the double contraction of two tensors \( F^a_b F^b_a \) is a tensor. Its rank is 0 because we are contracting all the indices, and we see that the explicit expression is a scalar.

\[ F^b_a F^a_b = F^1_b F^2_0 + F^2_1 F^3_0 = -B_y B_z. \]

Problem 2:

a) In tensor notation
\[ (\mathbf{A} \times \mathbf{B})(\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B})_i (\mathbf{C} \times \mathbf{D})^i = \epsilon_{ijk} A^j B^k \epsilon^{lm} C_l D_m = \epsilon_{ijk} \epsilon^{lm} A^j B^k C_l D_m = (\delta_j^l \delta_k^m - \delta_j^m \delta_k^l) A^j B^k C_l D_m = A^j B^k C_l D_m - A^m B^l C_l D_m = A^l C_l B^m D_m - A^m D_m B^l C_l = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}). \]

b) We get
\[ (\mathbf{A} \times \mathbf{B})(\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{B}) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{B} \cdot \mathbf{A}) = A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2, \]

which vanishes if \( \mathbf{A} \) and \( \mathbf{B} \) are parallel or antiparallel to each other.

Problem 3:

a) In this case \( \rho(\mathbf{r}) = Q \delta(x - 2) \delta(y + 1) \delta(z - 1). \)

b) In this case \( \rho(\mathbf{r}) = \frac{Q \delta(r - \sqrt{5}) \delta(\phi + 0.147 \pi) \delta(z - 1)}{r}. \)
c) In this case we can work in cylindrical coordinates so that \( \rho(\mathbf{r}) = \frac{Q\delta(r-R)\delta(z)}{2\pi r} \), or in spherical with \( \mathbf{r} = (r, \theta, \phi) \) for which \( \rho(\mathbf{r}) = \frac{Q\delta(r-R)\delta(\theta-\pi/2)}{2\pi r^2} \), or in spherical with \( \mathbf{r} = (r, \cos \theta, \phi) \) for which \( \rho(\mathbf{r}) = \frac{Q\delta(r-R)\delta(\cos \theta)}{2\pi r^2} \).

**Problem 4:**

a) We see that

\[
2\hat{z} - 4\hat{y} = \hat{x} \times \mathbf{B} = B_y\hat{z} - B_z\hat{y},
\]

\[
4\hat{x} - \hat{z} = \hat{y} \times \mathbf{B} = B_x\hat{x} - B_z\hat{z},
\]

\[
\hat{y} - 2\hat{x} = \hat{z} \times \mathbf{B} = B_x\hat{y} - B_y\hat{x}.
\]

Then, we see that \( B_x = 1, B_y = 2, \) and \( B_z = 4, \) i.e.,

\[
\mathbf{B} = (1, 2, 4).
\]

b) In tensor form

\[
\nabla.(\mathbf{u} \times \mathbf{v}) = \partial_a \epsilon^{abc} u_b v_c = \epsilon^{abc} v_c \partial_a u_b + \epsilon^{abc} u_b \partial_a v_c =
\]

\[
\epsilon^{abc} v_c \partial_a u_b + \epsilon^{bca} u_b \partial_a v_c = v_c \epsilon^{abc} \partial_a u_b - u_b \epsilon^{bca} \partial_a v_c = \mathbf{v}.(\nabla \times \mathbf{u}) - \mathbf{u}.(\nabla \times \mathbf{v}) = 0.
\]