Problem 1:

a) We have to solve Laplace’s equation because the region is free of charge.

b) The equation is

\[
\frac{\partial^2 \Phi(x, y)}{\partial x^2} + \frac{\partial^2 \Phi(x, y)}{\partial y^2} = 0.
\]  

(1)

c) A general solution will have the form:

\[
\Phi(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b},
\]  

(2)

It satisfies Eq.(1) and the b.c.’s for \( \Phi = 0 \). The coefficient \( A_n \) is determined from the b.c. at \( x = a \):

\[
V \sin \frac{\pi y}{b} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi a}{b},
\]  

(3)

from orthogonality of the sines we see that \( A_n = 0 \) for all \( n \neq 1 \) and

\[
A_1 = \frac{V}{\sinh \frac{\pi a}{b}}.
\]  

(4)
Then,

\[
\Phi(x, y) = \frac{V}{\sinh \frac{\pi a}{b}} \sin \frac{\pi y}{b} \sinh \frac{\pi x}{b}.
\] (5)

d) At the center of the region

\[
\Phi\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{V}{\sinh \frac{\pi a}{2b}} \sinh \frac{\pi a}{2b}.
\] (6)

**Problem 2:**

a) We need to solve Laplace’s equation in two different regions defined by the charged spherical surface. We cannot work in a single region because there is charge at \( r = a \) and Laplace’s equation is not valid there.

b) I expect to obtain the solution in terms of powers of \( r \) and Legendre polynomials because the boundary conditions are defined on a sphere and there is azimuthal symmetry.

c) In region I \((r \leq a)\) I propose:

\[
\Phi^I(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta),
\] (6)

we set to zero the coefficient of negative powers of \( r \) since the potential cannot diverge at \( r = 0 \). In region II \((r \geq a)\) I propose:

\[
\Phi^{II}(r, \theta) = \sum_{l=0}^{\infty} B_l r^{l+1} P_l(\cos \theta),
\] (7)

where the coefficients of positive powers of \( r \) have been set to zero because the potential has to vanish as \( r \to \infty \).

d) In order to determine the two sets of undetermined coefficients \( A_l \) and \( B_l \) I need two boundary conditions. We know that at \( r = a \) the potential has to be continuous then:

\[
\Phi^I|_{r=a} = \Phi^{II}|_{r=a}.
\] (8)
We also know that the normal component of the electric field across a charged surface has a jump equal to \( \sigma/\varepsilon_0 \) where \( \sigma \) is the surface density of charge. In this case the normal to the surface is the radial component, then \( E_n = E_r = -\frac{\partial \Phi}{\partial r} \)
and the second boundary condition becomes:

\[
\frac{\partial \Phi_{II}}{\partial r} |_{r=a} - \frac{\partial \Phi_{I}}{\partial r} |_{r=a} = -\frac{\sigma_0 \cos \theta}{\varepsilon_0}.
\]  

(9)

d) From Eq.(8) we find that

\[
A_l = \frac{B_l}{a^{2l+1}}.
\]

(10)

And from Eq.(9) we obtain:

\[
\sum_{l=0}^{\infty} [-(l+1)\frac{B_l}{a^{l+2}} - lA_l a^{l-1}] P_l(\cos \theta) = -\frac{\sigma_0 \cos \theta}{\varepsilon_0}.
\]

(11)

Notice that \( \cos \theta = P_1(\cos \theta) \). Thus multiplying both sides of Eq.(11) by \( P_m(\cos \theta) \) and integrating over \( \cos \theta \) in the interval \([-1,1]\) we obtain:

\[
-(m+1)\frac{B_m}{a^{m+2}} - mA_m a^{m-1} = -\frac{\sigma_0 \delta_{m,1}}{\varepsilon_0}.
\]

(12)

Replacing Eq.(10) in Eq.(12) we get:

\[
(2m+1)\frac{B_m}{a^{m+2}} = \frac{\sigma_0 \delta_{m,1}}{\varepsilon_0}.
\]

(13)

Then, if \( m \neq 1 \), \( A_m = B_m = 0 \). If \( m = 1 \)

\[
B_1 = \frac{\sigma_0 a^3}{3\varepsilon_0},
\]

(14)

and

\[
A_1 = \frac{\sigma_0}{3\varepsilon_0}.
\]

(15)

Replacing in Eq.(6) and (7) we obtain:

\[
\Phi_I(r, \theta) = \frac{\sigma_0}{3\varepsilon_0} r \cos \theta,
\]

(16)

and

\[
\Phi_{II}(r, \theta) = \frac{\sigma_0 a^3}{3\varepsilon_0 r^2} \cos \theta.
\]

(17)
Problem 3:

a) We can expand $f(x)$ in terms of Legendre polynomials because they form a set of orthogonal functions in the interval $[-1,1]$ in which $f(x)$ is defined.

![Graph showing the function $f(x)$ with key points at $-1$, $a$, and $1$.]

A formal expression for the expansion is given by

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x). \quad (18)$$

b) Using orthogonality of the Legendre polynomials we find that

$$a_l = (2l + 1) \int_a^1 P_l(x) dx. \quad (19)$$

To find the first 3 coefficients we need to set $l = 0, 1, \text{ and } 2$ and perform the integral. We obtain:

$$a_0 = (1 - a), \quad (20)$$

$$a_1 = \frac{3}{2}(1 - a^2), \quad (21)$$

$$a_2 = \frac{5a}{2}(1 - a^2). \quad (22)$$
c) Using the hint we can easily solve the integral in Eq.(19) and we obtain:

\[ a_l = (2l + 1) \int_a^1 P_l(x) \, dx = (P_{l+1}(x) - P_{l-1}(x))|_a^1 = P_{l-1}(a) - P_{l+1}(a), \]  

(23)

where we have used that \( P_l(\pm 1) = 1 \). Then we obtain that

\[ a_1 = P_0(a) - P_2(a) = \frac{3}{2} (1 - a^2), \]  

(24)

and

\[ a_2 = P_1(a) - P_3(a) = \frac{5a}{2} (1 - a^2). \]  

(25)

d) Now let’s calculate

\[ \int_{-1}^1 [f(x)]^2 \, dx = \int_{-1}^1 \sum_{l,m} a_la_m P_l(x)P_m(x) \, dx = \sum_{l,m} a_la_m \frac{2}{2l + 1} \delta_{l,m} = 2 \sum_{l=0}^{\infty} \frac{a^2}{2l + 1}. \]  

(26)

Using the result obtained in Eq.(23) valid for \( l > 0 \) and Eq.(20) we find that

\[ \int_{-1}^1 [f(x)]^2 \, dx = 2(1 - a^2) + \sum_{l=1}^{\infty} \frac{(P_{l-1}(a) - P_{l+1}(a))^2}{2l + 1}. \]  

(27)

**Problem 4:**

a) We use the principle of superposition to write the total potential as the sum of the potential of the individual charges \( q_i \) located at \( r_i \) given by \( \Phi_i(r) = \frac{q_i}{4\pi\varepsilon_0 |r-r_i|} \). Then,
\[ \Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{1}{|\mathbf{r} - \mathbf{r}_2|} - \frac{1}{|\mathbf{r} - \mathbf{r}_3|} - \frac{1}{|\mathbf{r} - \mathbf{r}_4|} \right) \]  

(28)

where the vectors \( \mathbf{r}_i \) are indicated in the figure.

b) The problem does not have azimuthal symmetry and thus, the potential has to be expanded in terms of the spherical harmonics. Since

\[ \frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r^l}{(2l + 1)r_{l+1}} Y_l,m(\theta, \phi) Y_l,-m(\theta', \phi'), \]  

(29)

we can replace Eq.(29) in Eq.(28) using that \( \theta_i = \pi/2 \) for all the charges and \( \phi_1 = 0, \phi_2 = \pi/2, \phi_3 = \pi, \) and \( \phi_4 = 3\pi/2. \) Since for \( r > a, r_< = a \) and \( r_> = r \) we obtain,

\[ \Phi(r, \theta, \phi) = \frac{q}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{a^l}{(2l + 1)r_{l+1}} Y_l,m(\theta, \phi)[Y_l,-m(\pi/2, 0) + Y_l,m(\pi/2, \pi) - Y_l,-m(\pi/2, \pi) - Y_l,-m(\pi/2, 3\pi/2)]. \]  

(30)