Problem 4 - 4.1.11:

We need to prove that $K_{ij}$ is a tensor knowing that $A^{jk}$ and $B_i^k$ are tensors.

In $S$:

$$K_{ij}A^{jk} = B_i^k. \quad (1)$$

In $S'$:

$$K'_{ij}A'^{jk} = B_i'^k. \quad (2)$$

Since we know that $B_i^k$ is a tensor

$$B_i^k = \frac{\partial x^l}{\partial x'^i} \frac{\partial x'^k}{\partial x^m} B_i^m, \quad (3)$$

Using Eq.(1) to replace $B_i^m$ in Eq.(3) we obtain:

$$B_i^k = \frac{\partial x^l}{\partial x'^i} \frac{\partial x'^k}{\partial x^m} A^{lm} \cdot K_l^r. \quad (4)$$

Since $A$ is a tensor we know that

$$A^{lm} = \frac{\partial x^r}{\partial x'^j} \frac{\partial x'^m}{\partial x^k} A'^{jk}. \quad (5)$$

Replacing Eq.(5) in Eq.(4) we obtain:

$$B_i^k = \frac{\partial x^l}{\partial x'^i} \frac{\partial x'^k}{\partial x^m} K_l^r \frac{\partial x^r}{\partial x'^j} \frac{\partial x'^m}{\partial x^k} A'^{jk}. \quad (6)$$

Comparing Eq.(2) with Eq.(6) we get:

$$K'_{ij}A'^{jk} = \frac{\partial x^l}{\partial x'^i} \frac{\partial x'^k}{\partial x^m} K_l^r \frac{\partial x^r}{\partial x'^j} \frac{\partial x'^m}{\partial x^k} A'^{jk}. \quad (7)$$

Rearranging terms in Eq.(7) we get:

$$(K'_{ij} - \frac{\partial x^l}{\partial x'^i} \frac{\partial x'^k}{\partial x^m} K_l^r A^{jk})A'^{jk} = 0. \quad (7)$$

Since $A$ is a non-zero arbitrary tensor we know that its coefficient has to vanish to satisfy Eq.(7) then:

$$K'_{ij} = \frac{\partial x^l}{\partial x'^i} \frac{\partial x^r}{\partial x'^j} K_l^r, \quad (8)$$

where we have used that $\frac{\partial x'^k}{\partial x^m} \frac{\partial x^m}{\partial x^r} = 1$. Eq.(8) shows that $K$ transforms like a covariant tensor of rank 2, then, $K_{ij}$ is a tensor.