Faraday's law of induction

Faraday did experiments involving a circuit in a time-dependent magnetic field. He observed that in the circuit a current was induced when:

1. the current in another nearby circuit was turned on and off
2. the second circuit with a current was moved relative to the first
3. a permanent magnet was moved in and out of the circuit

The "magnetic flux" is defined as:

\[ \Phi = \oint \mathbf{B} \cdot \mathbf{n} \, d\mathbf{a} \]

The "electromotive force" is defined as

\[ \mathcal{E} = \oint \mathbf{E} \cdot \, d\mathbf{r} \]
Faraday's experiments led to:

\[ E = -k \frac{dF}{dt} \]

the induced current opposes the change of flux.

Example:

If \( B \) increases with time, then \( \frac{dF}{dt} > 0 \). Thus, the induced current is circulating as shown, because it generates a \( B \) pointing as shown.

In our unit system (SI), \( k = 1 \).

\( \vec{E} \) is the electric field at \( \vec{d} \) in its "rest frame of coordinates".

The very nice discussion about Galilean transformations is left to the students to read. It addresses the fact that in the previous page, example (2), which of the two circuits is moving should be irrelevant.
Let us write Faraday's law in differential form.

\[ \oint_{C} \mathbf{E} \cdot d\mathbf{l} = \int_{S} (\nabla \times \mathbf{E}) \cdot \mathbf{n} \, d\mathbf{a} = -\frac{d}{dt} \int_{S} \mathbf{B} \cdot \mathbf{n} \, d\mathbf{a} = \int_{S} \left( \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \mathbf{n} \, d\mathbf{a} \]

Stokes' theorem

Thus:

\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \]

Generalization of \( \nabla \times \mathbf{E} = 0 \) to time dependent fields.

If circuit is held fixed in the chosen reference frame i.e. "S" and "C" are not charging.

We leave sections 5.16 and 5.17 for the students to handle.
5.18 A. Skin depth, Eddy currents, ...

\[
\begin{align*}
\text{empty space} & \quad z > 0 \\
\text{medium with uniform conductivity } & \quad \sigma \\
\text{and permeability } & \quad \mu \\
\text{and } x \geq 0
\end{align*}
\]

At the surface we introduce a magnetic field in the \( x \) direction:

\[ H_x(t) = H_0 \cos(\omega t) \]

that is not changing in space but changes in time.

What happens inside the medium?

From \( \nabla \times \mathbf{H} = \mathbf{J} \), we can say that in the medium:

\[ \nabla \times \mathbf{B} = \mu \mathbf{J} \]

since \( \mu \) in the medium is constant.

In addition, let us assume Ohm's law to be valid:

\[ \mathbf{J} = \sigma \mathbf{E} \]

Then:

\[ \nabla \times \mathbf{B} = \mu \sigma \mathbf{E} \]

\[ \nabla \times (\nabla \times \mathbf{A}) = \mu \sigma \mathbf{E} \]

\[ \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = 0 \text{ in Coulomb's gauge} \]

\[ \nabla \times \mathbf{E} = \nabla \left( \frac{\partial \mathbf{A}}{\partial t} \right) \]

From Faraday's law:
Then:

\[ \nabla^2 \vec{A} = \mu_0 \sigma \frac{\partial \vec{A}}{\partial t} \]  

(5.169)

Note: In

\[ \nabla \times \vec{E} = \nabla \times \left( \frac{\partial \vec{A}}{\partial t} \right) \]

we could have said

\[ \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi \]

To write \( \vec{E} = -\frac{\partial \vec{A}}{\partial t} \) we need to assume also:

\[ \text{[negligible free charge] } \]
\[ \text{[i.e. time varying B] } \]
\[ \text{[only source of } \vec{E} \text{].} \]

What boundary conditions?

From (5.86) and (5.87), and knowing that there is no surface current density, then:

\[ (\vec{B}_2 - \vec{B}_1) \cdot \vec{n} = 0 \]
\[ \vec{n} \times (\vec{H}_2 - \vec{H}_1) = 0 \]

Since at \( z < 0 \) there is no \( B_z \), then inside the medium the \( z \) component is also 0, from \( B_z \vec{n} = \vec{B}_1 \cdot \vec{n} \)

\[ B_z = B_1 \]

From \( \vec{n} \times \vec{H}_2 = \vec{n} \times \vec{H}_1 \):
\[ \vec{H}_1 = (H_x, 0, 0) \]
\[ \vec{m} = (0, 0, 1) \]

Then, \[ \vec{m} \times \vec{H}_1 = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & 1 \\ H_x & 0 & 0 \end{vmatrix} = H_x \mathbf{e}_y \] (outside medium, \( z < 0 \))
\[ \vec{m} \times \vec{H}_2 = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & 1 \\ H_x^2 & H_y^2 & H_z^2 \end{vmatrix} = -H_y^2 \mathbf{e}_x + H_x^2 \mathbf{e}_y \] (\( z > 0 \))

Then: \[ H_y^2 = 0 \text{ for } z > 0 \text{ (actually } z = 0^+ \) \]
\[ H_x^2 = H_x = H_0 \cos \omega t \text{ (at } z = 0^+). \]

Once we know that in the medium \( z > 0 \) the boundary condition is such that \( \vec{H} \) at \( z = 0^+ \) points along \( x \), then \( \vec{H} \) has only an \( x \)-component in the entire \( z > 0 \) domain because \[ \nabla^2 \vec{A} = \frac{\mu_0}{\sigma} \frac{\partial \vec{A}}{\partial t} \]
is linear. For instance, for the \( y \)-component
\[ \nabla^2 A_y = \mu_0 \sigma \frac{\partial A_y}{\partial t} \]
But at \( z = 0^+ \), \( A_z = H_0(t^{0+}) \)
\[ \nabla \times \vec{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & H_0(t^{0+}) \end{vmatrix} = \mathbf{e} \times H(z). \]

Then \( A_y(x) \) at the boundary is 0, and the simplest solution is \( A_y = 0 \) at any \( z > 0 \).
Since $A_z = \mu H(z) \gamma^x$, then

$$\nabla^2 A_z = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) H(z) \gamma^x \quad \text{and} \quad \frac{\partial (H(z) \gamma^x)}{\partial t} = \mu \frac{\partial A_z}{\partial t}$$

\[ \frac{\partial^2}{\partial z^2} H(z) = \mu \frac{\partial}{\partial z} \frac{\partial H(z)}{\partial t} \]

Let us try $H(z, t) = h(z) e^{-i \omega t}$

$$\frac{\partial H(z, t)}{\partial t} = -i \omega h(z) e^{-i \omega t}$$

Then:

$$\left( \frac{\partial^2}{\partial z^2} + i \omega \mu \sigma \right) h(z) = 0 \quad \text{(S.163)}$$

Let us try $h(z) = e^{ikz}$

$$\frac{\partial^2 h(z)}{\partial z^2} = (ik)^2 e^{ikz} = -k^2 e^{ik} \quad \text{(k can be complex, in principle)}$$

Then:

$$k^2 = i \mu \omega$$

$$k = \pm i \sqrt{\mu \omega} = \pm e^{i \pi / 4} \sqrt{\mu \omega}$$

$$= \pm e^{i \pi / 4} \sqrt{\mu \omega} = \pm \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \sqrt{\mu \omega}$$

$$= \pm \frac{(1 + i)}{\sqrt{2}} \sqrt{\mu \omega} \quad \text{(S.164)}$$

$$h(z) = e^{ikz} = e^{\pm i \left( \sqrt{\mu \omega} \frac{z}{\sqrt{2}} \right)}$$
\[ S = \sqrt{\frac{1}{m n}} \]

\[ h(z) = e^{\pm (i-1) \frac{z}{\delta}} \]

The solution with \((+\) is physical since \(h(z)\) can't grow with \(z\).
\((-\) is unphysical\).

\[ h(z) = e^{-2/\delta} \cdot e^{i \frac{z}{\delta}} \]

\[ H(z, t) = A e^{-2/\delta} e^{i(2/\delta - \omega t)} \]

In general we can have a constant in front. Simply asking that \(H(z=0^+, t) = H_0 e^{-i\omega t}\) we get
\[ A = H_0 \]

and then taking the real part:

\[ H(z, t) = H_0 e^{-2/\delta} \cos(2/\delta - \omega t) \] (5.166)

The magnetic field falls off exponentially in \(z\). Thus, it is confined to a depth \(\delta\).

\[ H = 0 \]

\[ \text{If } \delta \to \infty \text{ (perfect conductor)} \]

Then \(S \to 0 \) (no fields inside a perfect conductor.

\[ \text{Keep space with a } H = 0 \]
Since \( H \) varies with time, then from \( \nabla \times \mathbf{H} = \mathbf{J} \)

\[
\mathbf{J} = \sigma \mathbf{E}
\]

we can get the electric field:

\[
\nabla \times \mathbf{H} = \begin{vmatrix}
\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\
\nabla_x & \nabla_y & \nabla_z \\
H_x & 0 & 0
\end{vmatrix} = \hat{\mathbf{e}}_y \nabla_z H_x \hat{\mathbf{e}}_z - \hat{\mathbf{e}}_z \nabla_y H_x
\]

\[
= \frac{d}{dz} \left[ H_0 e^{i2/8 \cos \left( \frac{z}{8} - ut \right)} \right]
\]

\[
E_y = \frac{1}{\sigma} H_0 \frac{d}{dz} \left[ e^{-i2/8 \cos \left( \frac{z}{8} - ut \right)} \right]
\]

\[
= \frac{1}{\sigma} H_0 e^{-i2/8} \frac{d}{dz} \left[ e^{i2/8 \cos \left( \frac{z}{8} - ut \right)} \right]
\]

\[
= \frac{1}{\sigma} H_0 e^{-i2/8} \frac{d}{dz} \left[ \cos \left( \frac{z}{8} - ut \right) \right]
\]

\[
= \frac{1}{\sigma} H_0 e^{-i2/8} \left[ \frac{1}{i2/8} \sin \left( \frac{z}{8} - ut \right) \right]
\]

\[
= \frac{1}{\sigma} H_0 e^{-i2/8} \left[ \frac{1}{i2/8} \sin \left( \frac{z}{8} - ut \right) \right]
\]

\[
= \frac{1}{\sigma} H_0 e^{-i2/8} \left[ \frac{1}{i2/8} \sin \left( \frac{z}{8} - ut \right) + \frac{1}{i2/8} \sin \left( \frac{z}{8} - ut + \pi \right) \right]
\]

\[
= \frac{1}{\sigma} H_0 e^{-i2/8} \cos \left( \frac{z}{8} - ut + \pi \right)
\]

\[
E_y = \frac{1}{\sigma} H_0 e^{-i2/8} \cos \left( \frac{z}{8} - ut + \pi \right)
\]

\[
(5.167)
\]

If \( \omega \to 0 \), \( \delta \omega = \sqrt{\frac{2 \omega}{\mu_0}} \to 0 \), then, in the static limit, the electric field inside the conductor is 0, as expected, since a static magnetic field cannot generate an electric field. The magnetic field penetrates all the way, and the electric field cannot penetrate due to eddy currents.
The ratio \( \frac{E_y}{cB_x} = \frac{E_y}{c\mu H_x} = \frac{\mu_0 \omega B_0}{c\mu_0 H_0} = \frac{\mu_0 S}{c} \)

in practice this # is \( \ll 1 \)
so the fields are mainly magnetic.

**H field pointing**
along \( x \)

**E field pointing**
along \( y \) and
much smaller than \( B \).
Since \( J_y = 0 \), \( E_y \),
there is a small current, \( I_y \)

\[
[K_y(t)] = \int_0^\infty J_y(z,t)dz = \int_0^\infty \frac{\mu_0 S H_0 e^{-\frac{z}{\xi}}}{12} e^{\frac{z}{\xi}} e^{2\frac{t}{\xi}} \cos\left(\frac{\xi}{8} - ut + \frac{\pi}{4}\right) dz
\]

\[
= \frac{\mu_0 S H_0 8}{12} \int_0^\infty e^{-\frac{z}{\xi}} \cos\left(\frac{\xi}{8} - ut + \frac{3\pi}{4}\right) dz
\]

\[
\frac{\xi}{8} = \frac{\xi}{8} \quad \downarrow
\]

\[
= \frac{H_0 8}{12} \cdot \left(-\frac{1}{12} \cos t\right)
\]

\( \text{(next page)} \quad = -H_0 \cos t \)

These are the "eddy currents" that are generated in the metal. In chapter 8, the discussion continues.

Basically these currents tend to generate a magnetic field that opposes the external magnetic field. These currents also generate heat (called "induction heat").
\[
\int_0^\infty dx \ e^{-x} \cos(x-\alpha) = \int_0^\infty dx \ e^{-x} \left( \frac{e^{i(x-\alpha)} + e^{-i(x-\alpha)}}{2} \right) = \\
\frac{1}{2} e^{-i\alpha} \int_0^\infty dx \ e^{(-1+i)x} + \frac{1}{2} e^{i\alpha} \int_0^\infty dx \ e^{(-1-i)x} = \\
\frac{1}{2} e^{-i\alpha} \left. \frac{e^{(-1+i)x}}{(-1+i)} \right|_0^\infty + \frac{1}{2} e^{i\alpha} \left. \frac{e^{(-1-i)x}}{(-1-i)} \right|_0^\infty = \\
\frac{1}{2} e^{-i\alpha} \frac{1}{-1+i} + \frac{1}{2} e^{i\alpha} \frac{1}{-1-i} = \\
\frac{1}{2} e^{-i\alpha} \frac{-1}{\sqrt{2}} e^{\frac{i\pi}{4}} + \frac{1}{2} e^{i\alpha} \frac{-1}{\sqrt{2}} e^{\frac{-i\pi}{4}} = \\
\frac{1}{2} e^{-i\alpha} \frac{-i}{\sqrt{2}} e^{(\alpha - \frac{3\pi}{4})} + \frac{1}{2} e^{i\alpha} \frac{i}{\sqrt{2}} e^{(\alpha - \frac{\pi}{4})} = \\
\frac{-1}{2\sqrt{2}} e^{-i\omega t} + \frac{-1}{2\sqrt{2}} e^{i\omega t} = -\frac{1}{\sqrt{2}} \cos(\omega t)
\]
Summary

Consider a real metal with $\Sigma$ finite.

If $\omega = 0$ (static magnetic fields), then $S = \sqrt{\frac{2}{\mu_0 \sigma}} \to \infty$. The magnetic field can penetrate in the entire sample.

$E \propto S \omega = \sqrt{\frac{2\omega}{\mu_0 \sigma}} \to 0$

No electric field generated by static $B$ is expected.

If $\omega = \text{finite}$, then $S$ is finite as well, and $E$ is finite, thus $J$ is finite:

If $\omega \to \infty$, then $S \to 0$, and no magnetic field can penetrate. This can be achieved by $|E| \to \infty$, i.e. $|J| \to \infty$ at the surface.

Surface current prevents $B$ from penetrating.
What happens if conductor is perfect $\sigma \to \infty$?

For $\omega$ finite, $\sigma \to 0$ and the magnetic field does not penetrate. The electric field $E \sim \frac{2\omega}{\sigma}$ inside. For $\sigma = \infty$, $\omega$ finite, the magnetic field cannot penetrate the material.

$\sigma = \infty$
$\omega > 0$

The surface current induced by Faraday's law kills $B$ inside.

The case studied in the previous page is more realistic: all conductors have a finite $\sigma$ while $\omega$ can be truly tuned from non-zero to zero. Then, a static magnetic field is expected to penetrate into a metal. There will be a magnetization $\vec{M}$ but it will be small: $\vec{M} = \frac{\vec{B}_{\text{def}}}{\mu_0} - \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{H} = (\mu - \mu_{\infty}) \frac{\vec{B}}{\mu_0}$

and $\mu - \mu_{\infty}$ is almost 0 for a metal.
The finite case is also realistic in the following Gedanken experiment. We introduce a magnetic field \( B_{\text{ext}} \) inside a metal at high \( T \) where \( \gamma \) is not large. Then, we cool down and suppose we cross a phase transition to a state with a high \( \gamma \). In this case the magnetic field remains inside the metal, i.e. it is not expelled.

If the transition is to a superconductor, then the Meissner effect kicks in and \( B \) is expelled, but that effect is not contained in the Mx. Eqs.

In the plane \( \omega-1/\gamma \) the limit \( \omega \to 0, \gamma \to 0 \) is singular. If we fix \( \frac{1}{\gamma} = 0 \) and then \( \omega \to 0 \), the field does not penetrate the metal. If we fix \( \omega = 0 \) and then do \( \frac{1}{\gamma} \to 0 \), the field remains inside.

\[
\begin{align*}
\omega &= 0 \\
\frac{1}{\gamma} &= 0 \\
\text{gives } &\gamma = \infty
\end{align*}
\]

Since \( S \sim \frac{1}{\sqrt{\omega \gamma}} \), the lines of constant \( S \) are given by \( \frac{1}{\sqrt{\omega \gamma}} = \text{constant} \) or \( \omega = \frac{1}{A^2 \gamma} \). Examples are shown as dashed lines.