We want to connect the "micro" with the "macro" descriptions.

First, note that $\nabla \cdot \mathbf{B} = 0$ is always valid even at microscopic level. Then, at macro level too:

\[ \nabla \cdot \mathbf{B} = 0 \]

This means we can talk about a vector potential $\mathbf{A}(\mathbf{x})$ even in the presence of matter.

The many molecular magnetic moments $\mathbf{m}_i$ produce in average a macroscopic magnetization

\[ \mathbf{M}(\mathbf{x}) = \sum_i N_i \langle \mathbf{m}_i \rangle \]

average of $\mathbf{m}_i$ per small volume at point $\mathbf{x}$

$N_i$ number of molecules in small volume

Of course, we can also have in the system a macroscopic current density $\mathbf{J}(\mathbf{x})$.

This magnetic moment can have any origin, including the intrinsic spin of the electrons.
\[ \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \left[ \frac{\overrightarrow{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{\overrightarrow{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \right] d^3x' \]

This is using (5.55) and shifting coordinates.

Like done before when deriving (4.31), particularly using page 29 (bottom Eq.) we get:

\[ \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \left[ \frac{\overrightarrow{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} + \left( \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \right) \right] d^3x' \]

Now we use a vector identity:

\[ \nabla \times (\psi \vec{A}) = \psi (\nabla \times \vec{A}) + (\nabla \psi) \times \vec{A} \]

Consider \( \psi = \frac{1}{|\vec{x} - \vec{x}'|} \), \( \vec{A} = \overrightarrow{M}(\vec{x}') \). Then:

\[ -\overrightarrow{A} \times (\nabla \psi) = \overrightarrow{M} \times \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) = -\nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \times \overrightarrow{M} \]

\[ = -\nabla \times \left( \frac{\overrightarrow{M}}{|\vec{x} - \vec{x}'|} \right) + \frac{1}{|\vec{x} - \vec{x}'|} \left( \nabla \cdot \overrightarrow{M} \right) \]
The integral of the first term cancels

\[ \int \nabla \times \left( \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) d^3 x' \] since it reduces to a surface integral which vanishes if all functions are "well behaved" and localized.

Stokes theorem

\[ \oint \mathbf{\nabla} \times \mathbf{F} \cdot d\mathbf{s} = \iint_{S} \mathbf{\nabla} \cdot \mathbf{F} d^2 s \]

Then:

\[ \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \left( \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{\nabla' \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) d^3 x' \] (S.78)

Thus, it is as if the macroscopic medium provides an effective current density \( \mathbf{\nabla} \times \mathbf{M}(\mathbf{x}') \) that includes

If \( \mathbf{A} \) is like (S.32), which was derived in freespace, by just replacing \( \mathbf{J} \) by \( \mathbf{J}_{eff} \), then (S.26) will also hold if \( \mathbf{J} \to \mathbf{J}_{eff} \).
Then:

\[ \nabla \times \vec{B} = \mu_0 \left[ \vec{J} + \nabla \times \vec{M} \right] \]  

(5.80)

If \( \vec{B} \) and \( \vec{M} \) are combined to

\[ \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \]

analogous to \( \vec{D} \) in electrostatics

Then

\[ \nabla \times \vec{H} = \nabla \times \left( \frac{\vec{B}}{\mu_0} \right) - \nabla \times \vec{M} = \frac{\vec{J}}{\mu_0} \]

(5.80)

not effective but "material" or "free"

\[ \nabla \cdot \vec{H} = 0 \]

\( B \) = magnetic induction
\( H \) = magnetic field

As in the case of electrostatics, to complete the set of formulas for calculations we need a relation between \( \vec{H} \) and \( \vec{B} \). Assuming linearity and isotropy

\[ \vec{B} = \mu \vec{H} \]

\( \mu \) = magnetic permeability
The boundary conditions at an interface are:

From $\nabla \cdot \vec{B} = 0$, the integral over a volume $V$

$$\int_V \nabla \cdot \vec{B} \, d^3x' = 0 \quad \text{of course.}$$

But this is $\oint \vec{B} \cdot \vec{n} \, d\sigma$ due to divergence theorem.

Then:

$$\oint \vec{B} \cdot \vec{n} \, d\sigma = 0$$

Surface

Now, apply this to a small volume at the interface as

As $d \to 0$, the only integrals that matter are the top and bottom ones. They become:

$$\vec{(B_2 - B_4)} \cdot \vec{n} = 0 \quad \text{(5.86)}$$

(See text for details)
Now consider a small loop $\sigma$ in Fig 1.4.

We deduce
$$\nabla \times \mathbf{H} = \mathbf{J} \quad (5.82)$$

Apply row Stokes theorem
$$\int (\nabla \times \mathbf{F}).d\mathbf{a} = \int_{\partial \mathbf{S}} \mathbf{F}.d\mathbf{s}$$

$$\int \nabla \times \mathbf{H}.d\mathbf{a} = \int_{\partial \mathbf{S}} \mathbf{H}.d\mathbf{t} = \int_{\mathbf{S}} \mathbf{J}.d\mathbf{a} = k \mathbf{t}$$

"Side" contribution vanishes as $\sigma \to 0$

Then, we get:
$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{m} = k \mathbf{t}$$

Unit vectors $\mathbf{t}$:

shown. Note that
$$\mathbf{t} = \mathbf{x} \times \mathbf{m}$$

Surface current $k \mathbf{t}$:

units of $k$

surface area.

$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot (\mathbf{E} \times \mathbf{m}) = \mathbf{R} \cdot \mathbf{t} = \mathbf{E} \cdot \mathbf{R}$$

$$(\mathbf{E} \times \mathbf{E}) = 0 \quad \text{and} \quad \mathbf{E} \cdot (\mathbf{E} \times \mathbf{E}) = \mathbf{E} \cdot (\mathbf{E} \times \mathbf{E})$$
Then
\[ \mathbf{F} \cdot [\mathbf{H} \times (\mathbf{H}_1 - \mathbf{H}_2)] = \mathbf{F} \cdot \mathbf{R} \]

Then
\[ \mathbf{m} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{R} \] (S.87)

\( \mathbf{m} \) points from 1 to 2 i.e. it is \( \mathbf{m}_{12} \).

Alternatively, using \( \mathbf{B} = \mu \mathbf{H} \),
\[ \mathbf{B}_2 \cdot \mathbf{m} = \mathbf{B}_1 \cdot \mathbf{m} \] becomes
\[ \mathbf{H}_2 \cdot \mathbf{m} = \frac{\mu_1}{\mu} \mathbf{H}_1 \cdot \mathbf{m} \]

and
\[ \mathbf{m} \times (\frac{\mathbf{B}_2}{\mu_2} - \frac{\mathbf{B}_1}{\mu_1}) = \mathbf{R} \]
5.9 How do we solve these problems?

Basically we need to solve
\[ \nabla \cdot B = 0, \quad \nabla \times H = J \]
with some relation between \( B \) and \( H \), \( H = H(B) \).

(1) Always we can say \( B = \nabla \times A \)

plus \( \nabla \times H(\nabla \times A) = J \)

and there are then equations in \( A \) only.

For linear media, \( B = \mu H \), and then

\[ \nabla \times (\nabla \times A) = \frac{J}{\mu} \]

\( \mu \) could be

\[ \text{region 1} \quad \text{region 2} \]

If \( \mu \) is constant in some region, then in that region

\[ \frac{1}{\mu} \nabla \times (\nabla \times A) = \frac{1}{\mu} (\nabla (\nabla \cdot A) - \nabla^2 A) = \frac{J}{\mu} \]

Choosing \( \nabla \cdot A = 0 \), then \( -\nabla^2 A = \mu J \)

Of course, in addition we have to deal with boundary conditions.
(B) $\vec{J} = 0$; Magnetic Scalar potential

If in some region $\vec{J} = 0$, then $\nabla \times \vec{H} = 0$ and $\vec{H}$ can be written as $\vec{H} = -\nabla \Phi_m$ (similarly to $\vec{E} = -\nabla \Phi$).

If the medium is linear then:

$$\nabla \cdot \vec{B} = \nabla \cdot (\mu \vec{H}) = -\nabla \cdot (\mu \nabla \Phi_m) = 0$$

If $\mu$ is "permeability constant", then

$$\nabla^2 \Phi_m = 0 \text{ in each region}$$

[Typical example: medium in uniform external magnetic field]

(C) Hard Ferromagnets

Here a $\vec{M}$ that is considered fixed, i.e. independent of applied fields. We will assume $\vec{J} = 0$.

2 We will use (B) above since $\vec{J} = 0$.

$$\nabla \cdot \vec{B} = \nabla \cdot (\mu_0 (\vec{H} + \vec{M})) = 0$$

(5.81)
In addition (3) says \( H = -\nabla \Phi M \). Then if no currents
\[
\nabla \cdot (\mu_0 (\nabla \Phi M) + \overrightarrow{M}) = 0
\]
assumed \( \Phi M \) always constant, thus I drop it.
\[
\nabla \cdot \overrightarrow{M} = 0
\]
\[
+ \nabla^2 \Phi M = -\rho_m = \nabla \cdot \overrightarrow{M}
\]
def.

From previous experience in electrodynamics:
\[
\Phi M = -\frac{1}{4\pi} \int \frac{\nabla^2 \cdot M(x')}{|x-x'|} \, d^3x'
\]

Since
\[
\nabla^2 \Phi M = -\frac{1}{4\pi} \int \frac{\nabla^2 \cdot M(x')}{|x-x'|} \, d^3x' = -4\pi \int (x-x') \, d^3x' = \nabla \cdot M(x)
\]

\( \checkmark \)
If \( \vec{M} \) is "well-behaved and localized", then we can integrate by parts as done many times before:

\[
\nabla \cdot (f \vec{A}) = \nabla f \cdot \vec{A} + f (\nabla \cdot \vec{A})
\]

\[
\int \nabla \cdot (f \vec{A}) = \int \frac{(\nabla f) \cdot \vec{A}}{\vec{A} \cdot \nabla f} + \int f (\nabla \cdot \vec{A})
\]

\( \Rightarrow \)

Thus:

\[
\int f (\nabla \cdot \vec{A}) = -\int \vec{A} \cdot \nabla f
\]

Then, using \( \vec{A} = \vec{M} \) and \( f = \frac{1}{|\vec{x} - \vec{x}'|} \):

\[
\Phi_M = \frac{1}{4\pi} \int \vec{M}(\vec{x}') \cdot \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x' = \nabla \left( \frac{1}{|\vec{x} - \vec{x}'|} \right)
\]

\[
= -\frac{1}{4\pi} \nabla \cdot \int \frac{\vec{M}(\vec{x}') d^3x'}{|\vec{x} - \vec{x}'|}
\]

\( (5.98) \)

If there are no boundary surfaces!
Let us now assume "hard" ferromagnets: 
$M(\vec{x})$ inside a volume $V$ and 0 outside.

Then, in this case there is a surface charge density $\sigma_M$. Then, the solution will be the one before plus the analog of (1.23) for electrodynamics. We need to find the value of $\sigma_M$. The value can be obtained similarly as the derivation of (4.46)

\[ \sigma_M = \overline{\vec{m}} \cdot \overrightarrow{M} \]

Then:

\[ \overrightarrow{M}(\vec{x}) = \frac{1}{4\pi} \int \frac{[\nabla \cdot \overrightarrow{M}(\vec{x}')] d^3x'}{|\vec{x} - \vec{x}'|} \]

\[ + \frac{1}{4\pi} \oint \frac{\overrightarrow{m} \cdot \overrightarrow{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d\vec{a}' \]

(5.100)

We leave method C.Cb) "Vector potential" for the readers to discuss.
(students)