6.7 Conservation of Energy

For a single charge the "rate of doing work" is \( q \vec{v} \cdot \vec{E} \) with \( \vec{v} \) the velocity of the charge.

The magnetic field does not contribute because the magnetic force is perpendicular to \( \vec{v} \).

From (1.18) \( W = -q \int_{A}^{B} \vec{E} \cdot d\vec{l} \). If we write \( W = \int_{A}^{B} dW \)

then \( dW = -q \vec{E} \cdot d\vec{l} \)

and \( \frac{dW}{dt} = -q \vec{E} \cdot \frac{d\vec{l}}{dt} = -q \vec{E} \cdot \vec{v} \)

"rate of doing work"

See also page 40, 1st ed. 1.11

If there are many charges \( \frac{dW}{dt} = - \sum_{i} q_i \vec{E}_i \cdot \vec{v}_i \)

Note that (1.18) talks about the work done to move a charge from \( A \) to \( B \). Here we talk about the work done by the fields.

Thus, there is a sign change.

\[ \frac{dW}{dt} = \int d^3 x \frac{\vec{J}(\vec{x}) \cdot \vec{E}(\vec{x})}{V} \]

It is a convention of EM energy into mechanical or thermal energy.
Now let us consider Maxwell’s equations:

\[ (6.62) \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \]

\[ \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \]

Now use the vector identity

\[ \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \]

\[ \int_{V} \mathbf{E} \, d^3x = \int_{V} \left[ \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right] \]

\[ -\frac{\partial \mathbf{B}}{\partial t} \] from Maxwell's Eqs. \( (6.62) \)

Suppose the medium is linear, i.e.

\[ \mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu \mathbf{H} \]

Then

\[ \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D}) = \frac{\partial \mathbf{E} \cdot \mathbf{D}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = 2 \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \]

\[ \frac{1}{\varepsilon} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{\varepsilon} \frac{\partial \varepsilon \mathbf{E}}{\partial t} \]

and

\[ \frac{\partial}{\partial t} (\mathbf{B} \cdot \mathbf{H}) = \frac{\partial \mathbf{B} \cdot \mathbf{H}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t} = 2 \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \]
Then:
\[ \frac{\partial}{\partial t} \left[ \frac{1}{2} (E \cdot \vec{D} + B \cdot \vec{H}) \right] = \vec{E} \cdot \vec{\partial B} + \vec{H} \cdot \vec{\partial E} \]

and
\[ \int \frac{\partial}{\partial t} \vec{J} \cdot \vec{E} \, d^3x = \int \frac{\partial}{\partial t} \left[ \vec{\omega} + \nabla \times (\vec{E} \times \vec{H}) \right] \, d^3x \]

We call this \( \vec{S} \), the "Poynting vector".

Since the equation is valid for an arbitrary \( V \), then the integrands must be equal:

\[ \int \vec{J} \cdot \vec{E} \, d^3x = \vec{\omega} + \nabla \times \vec{S} \]

(6.108)

\[ \vec{S} = \vec{E} \times \vec{H} \]

\[ \vec{\omega} \] like a Continuity equation.

\[ \vec{\omega} \] is the total energy density.

In (4.89) we read \( W = \frac{1}{2} \int \vec{E} \cdot \vec{D} \, d^3x \)

which for a linear medium is
\[ W = \frac{\varepsilon}{2} \int \vec{E} \cdot \vec{E} \, d^3x \]

which for \( E = E_0 \) we derived

\[ W = \varepsilon \frac{E_0^2}{2} \] (1.54)
With regards to the magnetic component
(S,148) says
\[ W = \frac{1}{2} \int \vec{H} \cdot \vec{B} \; d^3x. \]

Then: \( \mu \) is the sum of both contributions.

Eq (6.108) (or its integral form) expresses
the "conservation of energy":

\[ \oint \left( \frac{d^3x}{V} \right) \frac{\partial \mu}{\partial t} = \text{time rate of change of electromagnetic energy in } V \]

\[ \frac{\partial}{\partial t} \left( \int \frac{d^3x}{V} \right) \]

\[ \oint \frac{d^3x}{V} \nabla \cdot \vec{S} = \oint \frac{\vec{S} \cdot \vec{n}}{A} \; da = \text{energy flowing out through the surface per unit time} \]

\( \left( \vec{S} \text{ has units } \frac{\text{energy}}{\text{area} \times \text{time}} \right) \)

\[ -\oint \frac{d^3x}{V} \left( \int \vec{J} \cdot \vec{E} \right) = \text{Work done by the fields on the source inside } V, \text{ per unit time} \]

As argued next this can be considered to be changes in the charges

Again, this is valid only if the medium is strictly linear.
\[ \overline{\mathbf{J} \cdot \mathbf{E}} \] represents a conversion of electromagnetic energy into mechanical or heat energy.

So \( -\int_{V} \overline{\mathbf{J} \cdot \mathbf{E}} \, d^{3}x \) is energy given to the charges and currents (causing charges)

Thus:

\[ \frac{dE_{\text{mechanical}}}{dt} = \int_{V} \overline{\mathbf{J} \cdot \mathbf{E}} \, d^{3}x \]

Total energy of the particles inside \( V \) can be "kinetic," "internal," etc.

Let us assume particles do not move "in" and "out" of \( V \).

Then, for the combined system (fields + charges):

\[ \frac{dE_{\text{tot}}}{dt} = \frac{d}{dt} (E_{\text{mech}} + E_{\text{field}}) = -\int_{V} d^{3}x \nabla \cdot \mathbf{S} \]

\[ = \int_{\partial V} \mathbf{S} \cdot d\mathbf{a} \quad \text{(6.111)} \]

Note that in the vacuum:

If spherical waves are taking energy out of \( V \), note that \( \mathbf{S} \cdot d\mathbf{a} \) is \( \mathbf{e} \), thus \( \mathbf{e} \) in front needed to account for energy reduction.
\[ E_{\text{field}} = \int_{V} \mu_{0} d^{3}x = \int_{V} \frac{1}{2} \left( \vec{E}, \vec{B}, \vec{F} \right) d^{3}x = \]
\[ \text{if in vacuum} \quad \epsilon_{0} \vec{E} = -\frac{\vec{B}}{\mu_{0}} \]
\[ = \left( \frac{1}{2} \left( \epsilon_{0} \vec{E}, \vec{E} + \frac{1}{\mu_{0}} \vec{B}, \vec{B} \right) \right) d^{3}x = \]
\[ \mu_{0} \epsilon_{0} = \frac{1}{c^{2}} \]
\[ = \frac{1}{2} \epsilon_{0} \int_{V} \left( \vec{E}^{2} + c^{2} \vec{B}^{2} \right) d^{3}x \quad (6.111) \]

Note: (6.111) \[ \frac{d}{dt} (E_{\text{mech}} + E_{\text{field}}) = -\oint_{S} \vec{n} \cdot \vec{s} \, da \]

can be visualized in terms of charges and photons:

\[ \bullet = \text{matter. It is assumed not to leave the volume.} \]
\[ \circ = \text{photons (inside and/or leaving)} \]

\( \vec{n} \cdot \vec{s} \) is like the momentum perpendicular to walls of the photons exiting the volume. And energy and mom. are linearly related. The \( (\cdot) \) indicates loss of energy in EM fields.

If photons leave, \( \vec{n} \cdot \vec{s} \) negative and \( -\vec{n} \cdot \vec{s} \) is a gain.