SOLUTION:

a) We need to express $\hat{e}'_i$ in cartesian coordinates.

$$\hat{e}'_1 = (\cos 30^\circ, -\sin 30^\circ) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad (1)$$

and

$$\hat{e}'_2 = (0, 1). \quad (2)$$

b) In order to find the vectors $\hat{e}''_i$ that form the contravariant basis we have to use that

$$\hat{e}'_i \cdot \hat{e}'_j = \delta_{ij}. \quad (3)$$

Let’s propose that $\hat{e}'^1 = (a, b)$ and $\hat{e}'^2 = (c, d)$. Then using the values obtained in Eq.(1) and (2) in Eq.(3) we can solve for $a$, $b$, $c$, and $d$.

$$0 = \hat{e}'^1 \cdot \hat{e}'_2 = b,$$

then $b = 0$,

$$1 = \hat{e}'^1 \cdot \hat{e}'_1 = \frac{a\sqrt{3}}{2},$$

then $a = \frac{2\sqrt{3}}{3}$. Then

$$\hat{e}'^1 = \left(\frac{2\sqrt{3}}{3}, 0\right), \quad (4)$$

and

$$1 = \hat{e}'^2 \cdot \hat{e}'_2 = d,$$

then $d = 1$,

$$0 = \hat{e}'^2 \cdot \hat{e}'_1 = \frac{c\sqrt{3}}{2} - \frac{1}{2},$$

then $c = \frac{2\sqrt{3}}{3}$. Then

$$\hat{e}'^2 = \left(\frac{\sqrt{3}}{3}, 1\right). \quad (5)$$

c) In $S'$ a generic vector $r'^i$ has coordinates $x'^i$ and in $S$ a generic vector $r^i$ has coordinates $x^i$. From the figure we see that

$$x^1 = x'^1 \cos \beta = x'^1 \cos 30^\circ = x'^1 \frac{\sqrt{3}}{2},$$

(6)
and

\[ x^2 = x'^2 - x'^1 \sin \beta = -x'^1 \sin 30^\circ + x'^2 = -x'^1 \frac{1}{2} + x'^2. \quad (7) \]

d) The easiest way to do this is remembering that a generic vector \( \mathbf{r}' \) can be written in terms of its covariant components in the contravariant basis as:

\[ \mathbf{r}' = x'_1 \hat{e}'^1 + x'_2 \hat{e}'^2, \quad (8) \]

and that it also can be written in terms of its contravariant components in the covariant basis then we have that

\[ \mathbf{r}' = x'_1 \hat{e}^1 + x'_2 \hat{e}^2 = x'^1 \hat{e}'^1 + x'^2 \hat{e}'^2. \quad (9) \]

Then using the cartesian expressions for \( \hat{e}'^1 \) and \( \hat{e}'^2 \) found in parts (a) and (b) we obtain:

\[ x'_1 \left( \frac{2\sqrt{3}}{3}, 0 \right) + x'_2 \left( \frac{\sqrt{3}}{3}, 1 \right) = x'^1 \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) + x'^2 (0, 1), \]

which gives us two equations:

\[ x'_1 \frac{2\sqrt{3}}{3} + x'_2 \frac{\sqrt{3}}{3} = x'^1 \frac{\sqrt{3}}{2} \quad (10) \]

and

\[ x'_2 = -x'^1 + x'^2. \quad (11) \]

Replacing (11) in (10) we obtain:

\[ x'_1 = x'^1 - x'^2 \frac{1}{2}. \quad (12) \]

e) Now we simply have to use Eq.(6) and (7). For \( \mathbf{p} \) I obtain:

\[ p^i = (2\frac{\sqrt{3}}{2}, -\frac{1}{2} + 3) = (\sqrt{3}, 2), \quad (13) \]

and for \( \mathbf{k} \)

\[ k^i = (-2\frac{\sqrt{3}}{2}, \frac{1}{2} + 2) = (-\sqrt{3}, 3). \quad (14) \]

f) Now we just need to plug the contravariant components of vectors \( p \) in Eqs.(11) and (12):

\[ p'_i = (2 - \frac{3}{2}, -\frac{1}{2} + 3) = \left( \frac{1}{2}, 2 \right), \quad (15) \]

and for vector \( k \):

\[ k'_i = (-2 - \frac{3}{2}, \frac{2}{2} + 2) = (-3, 3). \quad (16) \]

g) We know that in \( S \)

\[ \mathbf{p} \cdot \mathbf{k} = p_i k^i = p_i k_i = pk \cos \alpha, \quad (17) \]
Then
\[ \alpha = \cos^{-1} \frac{p \cdot k}{pk}. \]  \hspace{1cm} (18)

Let’s calculate the absolute values:
\[ p = \sqrt{3 + 4} = \sqrt{7}, \]  \hspace{1cm} (19)
and
\[ k = \sqrt{3 + 9} = 2\sqrt{3}. \]  \hspace{1cm} (20)

Then
\[ \alpha = \cos^{-1} \left( \frac{-3 + 6}{2\sqrt{21}} \right) = \cos^{-1} \left( \frac{3}{2\sqrt{21}} \right) = 70.89^\circ \]  \hspace{1cm} (21)

h) Now I have to repeat the same calculation but in \( S' \) where
\[ p' \cdot k' = p'_i k'^i = p'k' \cos \alpha', \]  \hspace{1cm} (22)

Then
\[ \alpha' = \cos^{-1} \frac{p' \cdot k'}{p'k'}. \]  \hspace{1cm} (23)

Let’s calculate the absolute values. Now I need to use the expression given for the contravariant components and the covariant components found in part (f):
\[ p' = \sqrt{\frac{1}{2} + 2 \times 3} = \sqrt{7}, \]  \hspace{1cm} (24)
which equals \( p \) as expected since the magnitude of a vector is a scalar and
\[ k' = \sqrt{(-3)(-2) + 3 \times 2} = 2\sqrt{3}, \]  \hspace{1cm} (25)
which, as expected, equals the magnitude \( k \). Then
\[ \alpha' = \cos^{-1} \left( \frac{-1 + 4}{2\sqrt{21}} \right) = \cos^{-1} \left( \frac{3}{2\sqrt{21}} \right) = 70.89^\circ. \]  \hspace{1cm} (26)

We see that \( \alpha' = \alpha \) since the angle between the two vectors is a scalar.

i) Since at each point \( (x_1, x_2) \) the function \( \Phi \) provides a scalar value we see that \( \Phi \) is a scalar function, i.e., a tensor of rank 0.

j) All we need to do is to write \( \Phi \) in terms of the contravariant variables in \( S' \). Thus, using Eq.(6) and (7) we obtain
\[ \Phi'(x'^1, x'^2) = \left( x'^1 \frac{\sqrt{3}}{2} \right)^2 - \left( -x'^1 \frac{1}{2} + x'^2 \right) = \frac{3}{4} (x'^1)^2 + \frac{x'^1}{2} - x'^2. \]  \hspace{1cm} (27)

k) Using the contravariant coordinates of \( p'^i \) provided I obtain:
\[ \Phi'(2, 3) = \frac{3}{4}(2)^2 + \frac{2}{2} - 3 = 3 + 1 - 3 = 1. \] (28)

1) Using the cartesian coordinates of \( p^i \) obtained in (c):

\[ \Phi(\sqrt{3}, 2) = 3 - 2 = 1, \] (29)

which is the same result obtained in (k) because the function is a scalar.

m) Now we need to calculate \( \partial_i \Phi \).

\[ \partial_i \Phi = \left( \frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial x_2} \right) = (2x_1, -1) \] (30)

n) The gradient of the scalar function \( \Phi \) is a vector and thus, it is a tensor of rank 1.

o) Now we need to calculate \( \partial'_i \Phi' \), i.e., the gradient of the function in \( S' \). The easiest way is to use the expression for the transformed function that we obtained in (j), then:

\[ \partial'_i \Phi' = \left( \frac{\partial \Phi'}{\partial x'^1}, \frac{\partial \Phi'}{\partial x'^2} \right) = \left( \frac{3}{2} x'^1 + \frac{1}{2}, -1 \right). \] (31)

Notice that the coordinates obtained in Eq.(31) covariant in \( S' \), i.e., it means that

\[ \partial'_i \Phi' = \left( \frac{3}{2} x'^1 + \frac{1}{2} \right) \hat{e}'^1 - \hat{e}'^2. \] (32)

p) Using the result obtained in (m) and the result obtained in (e) for the coordinates of \( p^i \) in \( S \) we obtain

\[ \partial_i \Phi(\sqrt{3}, 2) = (2\sqrt{3}, -1). \] (33)

q) Using the result obtained in (o) and the coordinates of \( p'^{n} \) in \( S' \) given in the problem we obtain

\[ \partial'_i \Phi'(2, 3) = \left( \frac{3}{2} + \frac{1}{2} \right) \hat{e}'^1 - 3 \hat{e}'^2 = \left( \frac{7}{2} \right) \hat{e}'^1 - \hat{e}'^2. \] (34)

r) The norm of \( \partial_i \Phi(\sqrt{3}, 2) \) in \( S \) is obtained as

\[ \sqrt{\partial_i \Phi(\sqrt{3}, 2) \partial_i \Phi(\sqrt{3}, 2)} = \sqrt{12 + 1} = \sqrt{13}. \] (35)

s) The norm of \( \partial'_i \Phi'(2, 3) \) in \( S' \) is obtained as

\[ \sqrt{\partial'_i \Phi'(2, 3) \partial'^{n} \Phi'(2, 3)}. \] (36)
I need to calculate the contravariant coordinates of \( \partial^i \Phi(\sqrt{3}, 2) \) so that we can write

\[
\partial^i \Phi'(x^1 = 2, x^2 = 3) = a \hat{e}_1' + b \hat{e}_2'.
\]

We know that as a vector

\[
\partial^i \Phi'(x^1, x^2) = \partial^i \Phi(x^1, x^2),
\]

which means that

\[
a \hat{e}_1' + b \hat{e}_2' = 2 \sqrt{3} \hat{e}_1 - \hat{e}_2.
\]

Using the expression for \( \hat{e}_i' \) in the cartesian system \( S \) that we found in part (a) Eq.(38) becomes:

\[
a \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) + b(0, 1) = (2 \sqrt{3}, -1).
\]

Then

\[
(a \frac{\sqrt{3}}{2}, -\frac{a}{2}) + b = (2 \sqrt{3}, -1),
\]

and we obtain that \( a = 4 \) and \( b = 1 \) then,

\[
\partial^i \Phi'(2, 3) = 4 \hat{e}_1' + \hat{e}_2'.
\]

Now replacing in Eq.(36) we obtain:

\[
\sqrt{\partial^i \Phi'(2, 3) \partial^i \Phi'(2, 3)} = \sqrt{\frac{7}{2} + (-1)(1)} = \sqrt{14} - 1 = \sqrt{13}.
\]

We see that the result is the same as the one obtained in (r) as expected since the norm of a vector is a scalar.

t) Now we need to construct the tensor \( T^{ij} = p^i k^j \) in \( S \). Thus, we use the coordinates for \( p^i = (\sqrt{3}, 2) \) and \( k^j = (-\sqrt{3}, 3) \) obtained in (d):

\[
T^{ij} = \begin{pmatrix}
  p^1 k^1 & p^1 k^2 \\
  p^2 k^1 & p^2 k^2
\end{pmatrix} = \begin{pmatrix}
  -3 & 3 \sqrt{3} \\
  -2 \sqrt{3} & 6
\end{pmatrix}.
\]

The trace of \( T \) is just \(-3+6=3\).

u) The easiest way to do this is to use the covariant and contravariant coordinates of \( p^i \) and \( k^j \) given in the problem and calculated in part (f) which means that \( p^i = (2, 3) \), \( k^j = (-2, 2) \), \( p'_i = (\frac{1}{2}, 2) \) and \( k'_j = (-3, 3) \) then:

\[
T'^{ij} = \begin{pmatrix}
  p'^1 k'^1 & p'^1 k'^2 \\
  p'^2 k'^1 & p'^2 k'^2
\end{pmatrix} = \begin{pmatrix}
  -4 & 4 \\
  -6 & 6
\end{pmatrix}.
\]

\[
T^{ij} = \begin{pmatrix}
  p'^1 k'^1 & p'^1 k'^2 \\
  p'^2 k'^1 & p'^2 k'^2
\end{pmatrix} = \begin{pmatrix}
  -3 & 3 \sqrt{3} \\
  -2 \sqrt{3} & 6
\end{pmatrix}.
\]

\[
T'^{ij} = \begin{pmatrix}
  p'^1 k'^1 & p'^1 k'^2 \\
  p'^2 k'^1 & p'^2 k'^2
\end{pmatrix} = \begin{pmatrix}
  -1 & 1 \\
  -4 & 4
\end{pmatrix}.
\]

\[
T'^{ij} = \begin{pmatrix}
  p'^1 k'^1 & p'^1 k'^2 \\
  p'^2 k'^1 & p'^2 k'^2
\end{pmatrix} = \begin{pmatrix}
  -6 & 6 \\
  -9 & 9
\end{pmatrix}.
\]

We see that the traces of \( T'^{ij}, T'^{ij}, T'^{ij} \) and \( T'^{ij} \) are 2, 9/2, 3, and 3. Only the traces of the mixed forms of the tensor have the same value as the trace of \( T \) in \( S \). This is because these are the only traces that are tensors obtained from the contraction of the indices of \( T \) as \( T^{ij} \) and \( T'^{ij} \). The traces of \( T'^{ij}, T'^{ij} \) are not tensors.